V. Dobrosavljević

Department of Physics and National High Magnetic Field Laboratory Florida State University, Tallahassee, FL 32306, USA



1.1 Why study metal-insulator transitions?

The metal-insulator transition (MIT) is one of the oldest, yet one of the fundamentally least understood problems in condensed matter physics. Materials which we understand well include good insulators such as silicon and germanium, and good metals such as silver and gold. Remarkably simple theories (Ashcroft and Mermin, 1976) have been successful in describing these limiting situations: in both cases low temperature dynamics can be well described through a dilute set of elementary excitations. Unfortunately, this simplicity comes at a price: the physical properties of such materials are extremely *stable*. They prove to be very difficult to manipulate or modify in order to the meet the needs modern technology, or simply to explore novel and interesting phenomena.



Fig. 1.1 Three basic mechanism for electron localization.

The situation is more promising in more complicated materials, where relatively few charge carriers are introduced in an otherwise insulating host. Several such systems have been fabricated even years ago, and some have, in fact, served as basic building blocks of modern information technology. The most familiar are, of course, the doped

semiconductors which led to the discovery of the transistor. More recent efforts drifted to structures of reduced dimensionality and devices such as silicon MOSFETs (metal-oxide-semiconductor field-effect transistors), which can be found in any integrated circuit.



Fig. 1.2 Quantum critical behavior near a metal-insulator transition. Temperature dependence of the resistance for different carrier concentrations is shown schematically in (a).Well defined metallic or insulating behavior is observed only at temperatures lower than a characteristic temperature $T < T^*$ that vanishes at the transition. At $T < T^*$, the system is in the "quantum critical region", as shown in (b). As the system crosses over from metal to insulator, the temperature dependence of the resistivity changes slope from positive to negative.

1.1.1 Why is the MIT an important problem?

In contrast to elemental materials, in systems close to the MIT the physical properties change dramatically with the variation of control parameters such as the carrier concentration, the temperature, or the external magnetic field. Such sensitivity to small changes is, indeed, quite common in any material close to a phase transition. In doped insulators this sensitivity follows from the vicinity to the metal-insulator transition. The sharp critical behavior is seen here only at the lowest accessible temperatures, because a qualitative distinction between a metal and an insulator exists only at T = 0(Fig. 1.2). Since the basic degrees of freedom controlling the electrical transport properties are electrons, and the transition is found at T = 0, quantum fluctuations dominate the critical behavior. The metal-insulator transition should therefore be viewed as perhaps the best example of a quantum critical point (QCP), a subject that has attracted much of the physicist's fancy and imagination in recent years (Sachdev, 2011). As near other QCPs, one expects the qualitative behavior here to display a degree of universality, allowing an understanding based on simple yet fundamental physical pictures and concepts. Before we understand the basic mechanisms and process that control this regime (Fig. 1.1) one can hardly hope to have control over material properties even in very simple situations.

1.1.2 Why is the MIT a difficult problem?

From the theoretical point of view the problem at hand is extremely difficult for reasons that are easy to guess. The two limits, that of a good metal and that of a good insulator, are very different physical systems, which can be characterized by very different elementary excitations. For metals, these are fermionic quasiparticles corresponding to electrons excited above the Fermi sea. For insulators, in contrast, these are long-lived bosonic (collective) excitations such as phonons and spin waves. In the intermediate regime of the metal-insulator transition, both types of excitations coexist, and simple theoretical tools prove of little help. Some of these conceptual difficulties are a general feature of quantum critical points. For QCPs involving spin or charge ordering, the critical behavior can be described by examining the order parameter fluctuations associated with an appropriate symmetry breaking. In contrast, the MIT is more appropriately described as a dynamical transition, and an obvious order parameter theory is not available. For all these reasons, the intermediate regime between the metal and the insulator has remained difficult to understand both from the practical and the conceptual point of view.

1.1.3 Disorder and complexity



Fig. 1.3 In the last few years, fascinating examples of complex ordering around the metal-insulator transition are starting to emerge, due to advances of both the experimental probes and the theoretical tools available. Left panel: Percolative conduction in the half-metallic ferromagnetic and ferroelectric mixture $(La, Sr)MnO_3$ (Park, Hur, Guha and Cheong, 2004). Central panel: Inhomogeneous charge distribution revealed by scanning tunneling microscope (STM) spectroscopy (Kohsaka, Iwaya, Satow, Hanaguri, Azuma, Takano and Takagi, 2004) on underdoped cuprate $Ca_{2-x}Na_xCuO_2Cl_2$. Right panel: Strikingly similar "stripe glass" ordering is observed in a computer simulation of an appropriate model (Reichhardt, Reichhardt and Bishop, 2005).

After more then sixty years of study, the subject of metal-insulator transitions has become a very wide and complicated field of research. In many complex materials, particularly in transition metal oxides and other strongly correlated systems,

the emergence of the metal-insulator transition is often accompanied by changes in magnetic or structural symmetry. In many such cases the transition is dominated by material-specific details, and it is often of first order, not a critical point. Somewhat surprisingly, the situation is in fact simpler in presence of sufficient amounts of disorder, where the metal and the insulator have the same symmetries and the transition is not associated with any uniform ordering. In such cases, strong evidence exists indicating that the transition is a genuine quantum critical point of a fundamentally new variety, with properties that most likely dominate the behavior of many materials.

At first glance disorder may be viewed simply as a nuisance and an inessential complication. In recent years, though, fascinating new evidence is emerging (Fig. 1.3) revealing that genuinely new phenomena (Miranda and Dobrosavljevic, 2005) arise in presence of disorder and impurities. Put quite simply, a given configuration of impurities may locally favor one or another of several competing phases of matter. In many instances this gives rise to strongly inhomogeneous states that feature an enormous number of low-lying metastable states - giving rise to quantitatively new excitation, slow relaxation, and glassy fluctuations and response. These phenomena often dominates the observable properties in many systems, ranging from colossal magnetoresistance (CMR) manganites and cuprates, to diluted magnetic semiconductors, and even Kondo alloys. Even more surprisingly, recent work suggest that a plethora of intermediate heterogenous phases may emerge between the metal and the insulator, possibly even in absence of disorder. Such *complexity* emerges as a new paradigm (Dagotto, 2005) of the metal-insulator transition region, most likely requiring a description in terms of probability distribution function functions (PDFs) rather then simple minded order parameters.

1.1.4 MIT in the strong correlation era

The subject of MITs came to a renewed focus in the last two decades, following the discovery of high temperature superconductivity, which triggered much activity in the study of "bad metals" (for a recent perspective, see Ref. (Basov and Chubukov, 2011)). Many of the materials in this family consist of transition metal or even rare earth elements, corresponding to compounds which are essentially on the brink of magnetism. Here, conventional approaches proved of little help, but recent research has lead to a veritable avalanche of new and exciting ideas and techniques both on the experimental and the theoretical front. In many ways, these developments have changed our perspective on the general problem of the metal-insulator transition, emphasizing the deep significance of the physics of strong correlation. A common theme for all these systems seems to be the transmutation of conduction electrons into localized magnetic moments (Anderson, 1978), a feature deeply connected to the modification of the fundamental nature of elementary excitations as the metal-insulator transition is crossed.

1.1.5 The scope of this overview

In this overview we will not discuss all the possible examples of metal-insulator transitions, but will focus on the basic physical mechanisms that can localize the electrons in absence of magnetic or charge ordering, and produce well defined quantum critical behavior. Our emphasis will be on results providing evidence that strong correlation physics dominates such quantum critical points, where physical pictures based on weak-coupling approaches prove insufficient or even misleading.

1.2 Basic mechanisms of metal-insulator transitions

What determines whether a material is a metal or an insulator? In most cases the answer is provided by simply examining the electronic band structure (Ashcroft and Mermin, 1976) and the pattern of chemical bonding of a given compound. At present, solid state physicists and quantum chemists have an impressive toolbox of theoretical methods to determine the band structure with sometimes surprising accuracy. Quite generally, one calculates all the accessible electronic levels (Slater, 1934) for the valence electrons in a solid and populates them according to the Pauli principle. If the highest occupied electronic state - the Fermi energy - is within a band gap, then the material is an insulator, since it takes a finite (generally large) energy to excite the electron to the lowest accessible state in order to carry electrical current. Otherwise, when electronic bands are partially filled, then we expect metallic behavior.

1.2.1 Band transitions

Can one induce a metal-insulator transition within this band theory picture? This is possible if the gap can be induced to open at the Fermi surface by rearranging the charge or the spin density of the electrons in the ground state, i.e. when the system undergoes an ordering transition. Typically, this corresponds to some kind of Fermi surface instability where a charge or spin density wave (Grüner, 2000) formation leads to unit cell doubling. An important early example was the Slater theory (Slater, 1951) of itinerant antiferromagnets, where the gap opens due to magnetic ordering. Such instabilities are also common in low dimensional solids such as organic charge-transfer salts (Grüner, 2000), leading to rich phase diagrams with many exotic properties. Such ordering transitions typically take place at finite temperature, and can often be successfully described using conventional approaches based on the band theory picture. These situations have attracted considerable attention in recent years, but we will not them discuss them further in this overview.

When does band theory work?. The success of the band theory approach was so impressive that already in 1930s Slater announced (Slater, 1934) that the solid state physics is a solved problem, and that we only need fast computers to accurately predict physical properties of any material. In the last few decades the computers did become amazingly fast - but in many interesting cases the band structure approach proved insufficient. When does that happen? To understand this important issue from a general point of view we need to recall the implicit assumptions of the band structure approach.

The band theory picture describes the dynamics of one electron moving through a solid, while the effects of all the other electrons is approximated by modifying the effective potential energy surface - the pseudopotential - on which it moves. This approximation is generally expected to be valid whenever the kinetic energy of the

electrons is dominant over the other energy scales in the problem. A most naive estimate would involve simply calculating the so-called r_s -number: $r_s = E_c/E_F$, where E_c is the average Coulomb energy per particle and E_F is the Fermi energy. The r_s number ranges between 3 and 5 even in good metals, and thus one would naively think that band theory should never work. However, one has to keep in mind the following important facts (Ashcroft and Mermin, 1976) that minimize the role of interactions:

- In metals screening reduces the magnitude of electron-electron and electronimpurity interactions to a significant degree.
- The largest part of the Coulomb energy (Hartree and exchange terms) contributes to redefining the pseudopotential, and only the "correlation" energy gives rise to many-body effects.
- The Pauli principle considerably restricts the phase space for electron-electron scattering, as described by Fermi liquid renormalizations.

As a result, the excitations in an electron gas can be viewed as a dilute collection of quasi-particle excitations, as described by the Landau's Fermi liquid theory (Landau, 1957; Landau, 1959). In a nutshell, a Fermi liquid is "protected" by a large kinetic energy scale of the electrons in their ground state – a direct result of their Fermi statistics. In good metals, the Fermi energy is typically in the electron-Volt range and the effects of electron correlations and impurity scattering can be treated as small perturbations (Abrikosov, Gor'kov and Dzyaloshinskii, 1975).

And...when does it fail?. In materials close to the metal-insulator transition the situations is very different. Here, the Fermi energy is typically small, and the "quantum protectorate" of the Pauli principle starts to weaken. This situation is found, for example, in

- Narrow band materials such as transition-metal oxide V_2O_3 .
- Doped semiconductors such as Si:P or diluted two-dimensional electron gases.
- Doped magnetic (Mott) insulators such as the famous high- T_c cuprate $La_{2-x}Sr_xCuO_4$.

In all these cases the potential energy terms coming from either residual electronelectron interactions or due to disorder (electron-impurity interaction) become comparable to the Fermi energy, and the ground state of the system can undergo a sudden and dramatic change - the electrons become bound or "localized". The material ceases to conduct although band theory does not predict any gap at the Fermi surface. In the following we briefly describe the early ideas on how this can take place, and discuss some general features of such quantum critical points.

1.2.2 Interactions: the Mott transition

Many insulating materials have an odd number of electrons per unit cell, thus band theory would predict them to be metals - in contrast to experiments. Such compounds (e.g. transition metal oxides) often have antiferromagnetic ground states, leading Slater to propose that spin density wave formation (Slater, 1951) is likely at the origin of the insulating behavior. This mechanism does not require any substantial modification of the band theory picture, since the insulating state is viewed as a consequence of a band gap opening at the Fermi surface. According to Slater (Slater, 1951), such insulating behavior should disappear above the Neel temperature, which is typically in the 10^2 K range. Most remarkably, in most antiferromagnetic oxides, clear signatures of insulating behavior persist at temperatures well above any magnetic ordering, essentially ruling out Slater's weak coupling picture.

What goes on is such cases was first clarified in early works by Mott (Mott, 1949) and Hubbard (Hubbard, 1963), tracing the insulating behavior to strong Coulomb repulsion between electrons occupying the same orbital. Within this picture, which is appropriate for narrow-band systems (Mott, 1990), the electrons tunnel between weakly hybridized atomic orbitals, as described by a Hubbard Hamiltonian

$$H_{HUB} = -\sum_{\langle ij\rangle\sigma} \left(tc^{\dagger}_{i\sigma}c_{j\sigma} + \text{h.c.} \right) + \sum_{j\sigma} \epsilon_j c^{\dagger}_{j\sigma}c_{j\sigma} + U \sum_j c^{\dagger}_{j\uparrow}c_{j\uparrow}c^{\dagger}_{j\downarrow}c_{j\downarrow}.$$
(1.1)

Here, the operator $c_{i\sigma}^{\dagger}$ creates an electron of spin σ in the *i*-th orbital, *t* is the tunneling element describing the inter-orbital hybridization, ϵ_j represents the corresponding site energy, and *U* describes the on-site Coulomb repulsion.



Fig. 1.4 Phase diagram for a fully frustrated half-filled Hubbard model calculated from DMFT theory. At low temperatures the Fermi liquid and the Mott insulating phases are separated by a first order transition line, and the associated coexistence region. Very recent work (Terletska, Vučičević, Tanasković and Dobrosavljević, 2011) established that at $T > T_c$ the intermediate metal-insulator crossover region show all the features expected for the quantum critical regime, including the characteristic scaling behavior for the family of resistivity curves.

When the lattice has integer filling per unit cell, then electrons can be mobile only if they have enough kinetic energy $(E_K \sim t)$ to overcome the Coulomb energy U. In the narrow band limit of $t \ll U$, the electrons do not have enough kinetic energy, and a gap opens in the single-particle excitation spectrum, leading to Mott insulating behavior. This gap $E_g \approx U - B$ (here $B \approx 2zt$ is the electronic bandwidth; z being the lattice coordination number) is the energy an electron has to pay to overcome the Coulomb repulsion and leave the lattice site. In the ground state, each lattice site is singly occupied, and the electron occupying it behaves as a spin 1/2 local magnetic moment. These local moments typically interact through magnetic superexchange interactions (Anderson, 1959) of the order $J \sim U/t_{ij}^2$, leading to magnetic ordering at temperatures of order $T_J \sim J$. The insulating behavior, however, is not caused by magnetic ordering, and will persist all the way to temperatures $T_{Mott} \sim E_g \gg T_J$. In oxides, $T_{Mott} \sim$ $E_g \sim 10^3 - 10^4 K$ is typically on the atomic (eV) scale, while magnetic ordering emerges at temperatures roughly an order of magnitude lower $T_J \sim 100 - 300K$.

We note that Mott's simple argument for the stability of the interaction-driven insulator does not directly rely on a periodicity of a lattice. Even if the site energies ε_i or the hopping elements t_{ij} are random variables of moderate variance, the Mott gap will persist, provided that the on-site repulsion U is large enough as compared to the typical kinetic energy. This is true, since the Mott gap essentially measures the energy for an electron to hop to the nearest lattice site - an inherently local process which does not depend much on long-range periodicity of the lattice. Precisely this behavior is what takes place in a doped semiconductor deep in the insulating phase, which should be viewed as a strongly disordered Mott insulator. Clear evidence for the correctness of this picture is provided by optical experiments (Thomas, Capizzi, DeRosa, Bhatt and Rice, 1981), which provide unambiguous evidence of the coexistence of the Mott gap at sufficiently low doping levels.

When the kinetic energy and the Coulomb interaction are comparable, the system finds itself in the vicinity of the Mott transition (Fig. 1.4). Experimentally, the bandwidth can often be controlled by modifying the orbital overlap t, thus the electronic bandwidth. In several transition metal oxides, for example, this is possible by applying external hydrostatic pressure. From the theoretical perspective, describing the vicinity of the Mott transition proves quite difficult due to the lack of a small parameter characterizing this nonperturbative regime. Still, after more then thirty years of work on the problem, several theoretical approaches have emerged, which provide the physical picture of the transition region. Early arguments of Mott and Hubbard make it clear that the gap will close for $U \leq B$, but the precise form of the critical behavior remained elusive.

Correlated metallic state close to the Mott transition. An important step in elucidating the approach to the Mott transition from the metallic side was provided by the pioneering work of Brinkmann and Rice (Brinkman and Rice, 1970). This work, which was motivated by experiments on the normal phase of ${}^{3}He$ (for a review, see: (Vollhardt, 1984)), predicted a strong effective mass enhancement close to the Mott transition. In the original formulation, as well as in its subsequent elaborations (slave boson mean-field theory (Kotliar and Ruckenstein, 1986), dynamical mean-field theory (Georges, Kotliar, Krauth and Rozenberg, 1996)), the effective mass is predicted to

Basic mechanisms of metal-insulator transitions 9

continuously diverge as the Mott transition is approached form the metallic side

$$\frac{n^*}{m} \sim (U_c - U)^{-1}.$$
(1.2)



Fig. 1.5 Clear evidence of strong mass enhancements can be seen in experiments on mono-layer He^3 films on graphite (Casey, Patel, Nyéki, Cowan and Saunders, 2003). In this system, the solid phase (Mott insulator) can be approached when the density is increased by the application of hidrostatic pressure.

A corresponding coherence (effective Fermi) temperature

$$T^* \sim T_F/m^* \tag{1.3}$$

is predicted above which the quasiparticles are destroyed by thermal fluctuations. As a result, one expects a large resistivity increase around the coherence temperature, and a crossover to insulating (activated) behavior at higher temperatures. Because the low temperature Fermi liquid is a spin singlet state, a modest magnetic field of the order

$$B^* \sim T^* \sim (m^*)^{-1} \tag{1.4}$$

is expected to also destabilize such a Fermi liquid and lead to large and positive magnetoresistance.

The physical picture of the Brinkmann-Rice seems to suggest that the Mott transition should be viewed as a quantum critical point, where powerlaw behavior of characteristic crossover scales is expected as the transition is approached. On the other

hand, the Mott transition discussed here describes the opening of a correlation-induced spectral gap in absence of magnetic ordering, i.e. within the paramagnetic phase. Such a phase transition is, therefore, not associated with spontaneous symmetry breaking associated with any static order parameter. Why should the phase transition then have any second-order (continuous) character at all?

The answer is that... in fact it does not! Later work (Moeller, Dobrosavljević and Ruckenstein, 1999; Park, Haule and Kotliar, 2008), which presented better descriptions of inter-site correlations, suggested that such a transition should generically have a (weakly) first order character, in agreement with early ideas of Mott (Mott, 1949). The effective mass, even if it does not exactly diverge at the transition, is still expected to be significantly enhanced in its close vicinity. Such behavior is indeed seen in various Mott systems (Georges, Kotliar, Krauth and Rozenberg, 1996) that have been studied, including bulk and two-dimensional He^3 liquids (Fig. 1.5), transition metal oxides, and organic charge-transfer salts.

Physical content of the effective mass enhancement. How should we physically interpret the large effective mass enhancement which is seen in all these systems? What determines its magnitude if it does not actually diverge at the transition? An answer to this important question can be given using a simple thermodynamic argument, which does not rely on any particular microscopic theory or specific model. In the following, we present this simple argument for the case of a clean Fermi liquid, although its physical context is, of course, much more general.

In any clean Fermi liquid (Abrikosov, Gor'kov and Dzyaloshinskii, 1975) the low temperature specific heat assumes the leading form

$$C(T) = \gamma T + \cdots, \qquad (1.5)$$

where the Sommerfeld coefficient

$$\gamma \sim m^*. \tag{1.6}$$

In the strongly correlated limit $(m^*/m \gg 1)$ this behavior is expected only at $T \lesssim T^* \sim (m^*)^{-1}$, while the specific heat should drop to much smaller values at higher temperatures where the quasiparticles are destroyed. Such behavior is indeed observed in many systems showing appreciable mass enhancements.

On the other hand, from general thermodynamic principles, we can express the entropy as

$$S(T) = \int_{0}^{T} dT \frac{C(T)}{T}.$$
 (1.7)

Using the above expressions for the specific heat, we can estimate the entropy around the coherence temperature

$$S(T^*) \approx \gamma T^* \sim O(1). \tag{1.8}$$

The leading effective mass dependence of the Sommerfeld coefficient γ and that of the coherence temperature T^* cancel out!

Let us now explore the consequences of the assumed (or approximate) effective mass divergence at the Mott transition. As $m^* \to \infty$, the coherence temperature $T^* \to 0+$, resulting in large residual entropy

$$S(T \longrightarrow 0+) \sim O(1). \tag{1.9}$$

We conclude that the effective mass divergence indicates the approach to a phase with finite residual entropy!

Does not this result violate the Third Law of Thermodynamics?! And how can it be related to the physical picture of the Mott transition? The answer is, in fact, very simple. Within the Mott insulating phase the Coulomb repulsion confines the electrons to individual lattice sites, turning them into spin 1/2 localized magnetic moments. To the extend that we can ignore the exchange interactions between these spins, the Mott insulator can be viewed as a collection of free spins with large residual entropy $S(0+) = R \ln 2$. This is precisely what happens within the Brinkmann-Rice picture; similar results are obtained from DMFT, a result that proves exact in the limit of large lattice coordination (Georges, Kotliar, Krauth and Rozenberg, 1996).



Fig. 1.6 Temperature dependence of entropy extracted from specific heat (inset) experiments (Flouquet, 2005) on several heavy-fermion materials. Essentially the entire doublet entropy $S = R \ln 2$ is recovered by the time the temperature has reacued $T^* \approx 10K$, consistent with a large mass enhancement $m^* \sim 1/T^*$.

In reality, the exchange interactions between localized spins always exist, and they generally lift the ground state degeneracy, restoring the Third Law. This happens below a low temperature scale T_J , which measures the effective dispersion of inter-site magnetic correlations (Moeller, Dobrosavljević and Ruckenstein, 1999; Park, Haule and Kotliar, 2008) emerging from such exchange interactions. In practice, this correlation

temperature T_J can be very low, either due to effects of geometric frustration, or additional ring-exchange processes which lead to competing magnetic interactions.

We conclude that the effective mass enhancement, whenever observed in experiment, indicates the approach to a phase where large amounts of entropy persist down to very low temperatures. Such situations very naturally occur in the vicinity of the Mott transition, since the formation of local magnetic moments on the insulating side gives rise to large amounts of spin entropy being released at very modest temperatures. A similar situation is routinely found (Stewart, 1984; Hewson, 1993; Flouquet, 2005) in the so-called "heavy fermion compounds(e.g. rare-earth intermetallics) featuring huge effective mass enhancements. Here, local magnetic moments coexist with conduction electrons giving rise to the Kondo effect, which sets the scale for the Fermi liquid coherence temperature $T^* \sim 1/m^*$, above which the entire free spin entropy $S(T^*) \sim R \ln 2$ is recovered (Fig. 1.6). This entropic argument is, in turn, used to experimentally prove the existence of localized magnetic moments within the metallic host.

We should mention that other mechanisms of effective mass enhancement have also been considered. General arguments (Millis, 1993) indicate that m^* can diverge when approaching a quantum critical point corresponding to some (magnetically or charge) long-range ordered state. This effect is, however, expected only below an appropriate upper critical dimension (Sachdev, 2011), reflecting an anomalous dimension of the incipient ordered state. In addition, this is mechanism produced by long wavelength order-parameter fluctuations, and is thus expected to contribute only a small amount of entropy per degree of freedom, in contrast to local moment formation. It is interesting to mention that weak-coupling approaches, such as the popular "on-shell" interpretation of the Random-Phase Approximation (RPA) (Ting, Lee and Quinn, 1975), often predict inaccurate or even misleading predictions (Zhang and Das Sarma, 2005) for the effective mass enhancement behavior. As discussed in Chapter 6, more accurate modern theories such as DMFT can be used to benchmark these and other weak coupling theories, and reveal the origin of some of the pathologies found when they are applied to strong coupling.

Finite temperature metal-insulator coexistence region. Since the Fermi liquid metal and the paramagnetic Mott insulator do not differ on symmetry grounds, there is no reason why these two phases cannot coexist in a finite range of parameter space. Indeed, recent theories (Georges, Kotliar, Krauth and Rozenberg, 1996) as well as several experimental studies (Limelette, Wzietek, Florens, Georges, Costi, Pasquier, Jerome, Meziere and Batail, 2003b; Limelette, Georges, Jerome, Wzietek, Metcalf and Honig, 2003a) in clean Mott system find such a metal-insulator coexistence region (Fig. 1.7) leading to a finite temperature first-order phase transition line. Experimentally, an appreciable drop of resistivity is seen as the system is driven though such a finite temperature Mott transition, which separates the Mott insulating state and the metallic (Fermi liquid) state. Similarly as in standard liquid-gas systems, the coexistence region, and the associated first-order line, terminate at the critical end-point at $T = T_c$. The corresponding critical behavior has been carefully studied in recent experiments (Limelette, Georges, Jerome, Wzietek, Metcalf and Honig, 2003a) on chromium-doped V_2O_3 , and was found to belong to the standard Ising universality class.

Basic mechanisms of metal-insulator transitions 13

Weak disorder near Mott transitions. All quantities display a discontinuity across any first-order phase transition, but this jump can be reduced in presence of impurities or disorder, which tend to locally favor one or the other phase. Well-known droplet arguments (Imry and Ma, 1975), then suggest that sufficiently strong disorder can completely suppress such a first-order transition (Berker, 1991), eliminating the finite temperature metal-insulator coexistence region. To illustrate this argument, consider an uncompensated doped semiconductor (M. A. Paalanen, 1991), where the bandwidth of the impurity band can be tuned by varying the donor concentration n. Assume that n is chosen to lie just below its critical value $n_c(T)$, so that even a small increase in n would drive the system through a Mott transition, leading to a large resistivity drop. Here, the reduced donor density $\delta n(T) = (n - n_c(T))/n_c(T)$ plays a role of the magnetic field in an ordinary ferromagnet, which can be used for $T < T_c$ to drive the system through a first-order transition where the magnetization jumps. A similar first order Mott metal-insulator transition is precisely what one would expect in our case as well - if only the donors were ordered with perfect periodicity. In fact, the original work of Mott (Mott, 1949) considered precisely such a scenario, where one imagines varying the donor concentration, while ignoring the local density fluctuations.



Fig. 1.7 Phase diagram of the organic salt $\kappa - (BETD - TTF)_2 Cu[N(CN)_2]Cl$ (Limelette, Wzietek, Florens, Georges, Costi, Pasquier, Jerome, Meziere and Batail, 2003b). In this material, increasing hydrostatic pressure broadens the electronic bandwith, favoring the metallic state. The first-order Mott transition extends at finite temperature up to the critical end-point at $T_c \approx 40K$. The corresponding coexistence region (shaded) displays hysteresis in transport.

Unfortunately, in a real material, donor ions assume random positions within the semiconducting host. A given region of size L can have a local concentration $n(L) \gtrsim n_c$,

favoring the formation of a metallic "droplet". For $\delta n_L = n(L) - n_c > 0$, the energy of the metallic phase will be lower by the amount

$$\Delta E_{M-I}(L) = \varepsilon \delta n_L, \qquad (1.10)$$

where ε is a constant measuring the density-dependent (free) energy difference between the metal and the insulator. Creating such a droplet will create a domain wall, which costs surface energy

$$E_s = \sigma L^{d-1},\tag{1.11}$$

where σ is proportional to the surface tension of the droplet. The droplet will be formed only if

$$\Delta E_{M-I}(L) > \sigma L^{d-1}. \tag{1.12}$$

Note, however, that $\Delta E_{M-I}(L) \sim \delta n_L$, which is a random quantity. To calculate the probability that a droplet of size L will be formed, we need to calculate the probability of a density fluctuation

$$\delta n_L > \sigma L^{d-1} / \varepsilon. \tag{1.13}$$

Assuming that the donor density fluctuations are uncorrelated on large enough scales, the probability distribution is given by the "central-limit theorem"

$$P(\delta n_L) \sim \exp\left\{-\frac{1}{2}\frac{\delta n_L^2}{L^d W^2}\right\},\tag{1.14}$$

where W is a constant measuring density fluctuations (disorder strength) on the microscopic scale. The probability that the droplet of size L will for will, therefore, be of order

$$P(L) \sim \exp\left\{-\frac{1}{2}\frac{\sigma^2 L^{2d-2}}{\varepsilon^2 L^d W^2}\right\} = \exp\left\{-\frac{1}{2}\frac{\sigma^2 L^{d-2}}{\varepsilon^2 W^2}\right\}.$$
 (1.15)

As we can see, for d > 2, large droplets are exponentially suppressed, i.e. their concentration is exponentially small, and for sufficiently weak disorder $(W \ll \sigma/\varepsilon)$, even the very small droplets are exponentially rare. The first-order transition remains sharp, at least at low enough temperatures.

But what happens when the temperature is increased and we approach the critical end-point at $T = T_c$? Here, we expect the droplet surface tension to decrease as a power of the correlation length $\xi \sim (T_c - T)^{-\nu}$ (in mean-field theory $\nu = 1/2$), so even weak disorder starts to have an appreciable effect. More precisely, here $\sigma(T) \sim \xi^{-3}$, and $\varepsilon(T) \sim \xi^{-1}$ (see, for example Ref. (Goldenfeld, 1992)), and even small droplets start to proliferate. The transition is then smeared down to temperatures such that $\varepsilon(T)W/\sigma(T) \sim O(1)$, i.e. $W \sim \xi^{-2} \sim (T_c - T)^{-2\nu}$. We conclude that the in presence of weak disorder, the critical temperature is depressed by

$$\delta T_c(W) = T_c(0) - T_c(W) \sim W^{1/2\nu}.$$
(1.16)

When disorder is sufficiently strong $(W \sim \sigma(0)/\varepsilon(0))$, the first order jump is completely eliminated at finite temperature, and only a smooth metal-insulator crossover remains. Such behavior is clearly seen in all standard doped semiconductors, where the positional disorder in donor or acceptor ions is so strong that no evidence of finite temperature Mott transition can be seen. A sharp distinction between the metal and the insulator is then found only at T = 0, where the transition reduces to the conventional quantum critical point. Of course, the resulting critical behavior is completely different from that of a clean system, where all the conduction electrons simultaneously turn into local magnetic moments at a well defined critical concentration n_c .

In the random case the local density undergoes strong spatial fluctuations. As a result, many regions form where the local density is much lower then the average. Here, one expects the electrons to undergo local Mott localization. In the remaining regions the local density is higher the average, and the electrons remain itinerant. This simple physical picture thus suggest a two-fluid behavior (Paalanen, Graebner, Bhatt and Sachdev, 1998) of conduction electrons and local magnetic moments - a situation very different then what one expects in a weakly disordered metal. Describing such disorder-enhanced strong correlation effects proves theoretically to be extremely difficult, since the theory must account for the effective interaction between such disorder-induced local moments and the remaining itinerant electrons. Developing such a theory should include an appropriate description of the corresponding Kondo screening processes in a disordered environment (Dobrosavljević, Kirkpatrick and Kotliar, 1992). This remains one of the most challenging open problems in this field, although important advances have recently been accomplished (Miranda and Dobrosavljevic, 2005) based on dynamical mean-field approaches.



Fig. 1.8 The simplest model for disorder-induced cluster states near first-order phase transitions is provided by the random-field Ising model (Imry and Ma, 1975). An illustration is provided by recent simulation results (Moreo, Mayr, Feiguin, Yunoki and Dagotto, 2000), which show how in d=2, stronger disorder (W=3 - pannel (a)) creates many small size clusters, while only few large ones remain for sufficiently weak disorder (W=1.5 - pannel (b)). Similar behavior is expected (Imry and Ma, 1975) near any disorder-smeared first order phase transition.

An especially interesting situation is found in d = 2, where both the bulk energy gain $\Delta E_{M-I}(L) = \varepsilon \delta n_L \sim LW$, and the surface energy $E_s = \sigma L$ both scale linearly

with the droplet size L, allowing for arbitrarily large droplets. The typical droplet size thus diverges, and the first-order transition is suppressed for arbitrarily weak disorder. This situation may be relevant for high mobility (weak disorder) two-dimensional electron gases (Abrahams, Kravchenko and Sarachik, 2001). Here, behavior reminiscent of a Mott transition $(m^* \sim (n - n_c)^{-1})$ seems to emerge (Kravchenko and Sarachik, 2004) only at T = 0, while only a smooth metal-insulator crossover persisting at finite temperatures. We should note, however, that the above expressions are valid only for sufficiently large droplets containing many impurities, such that Gaussian statistics applies. At weak disorder, the average distance between impurities ℓ is large, and the disorder can produce only droplets larger then a certain minimum size $L_{\min} \gg \ell$. We thus expect that reasonably large scale inhomogeneities should emerge (Fig. 1.8) when weak disorder is introduced near first-order metal-insulator transitions in d = 2.

Is Wigner crystallization a Mott transition in disguise?. The original ideas of Mott (Mott, 1949), who thought about doped semiconductors, envisioned electrons hopping between well localized atomic orbitals corresponding to donor ions. In other Mott systems, such as transition metal oxides, the electrons travels between the atomic orbitals of the appropriate transition metal ions. In all these cases, the Coulomb repulsion restricts the occupation of such localized orbitals, leading to the Mott insulating state, but it does not provide the essential mechanism for the formation such tightly bound electronic states. The atomic orbitals in all these examples result from the (partially screened) ionic potential within the crystal lattice.

The situation is more interesting if one considers an idealized situation describing an interacting electron gas in absence of any periodic (or random) lattice potential due to ions. Such a physical situation is achieved, for example, when dilute carriers are injected in a semiconductor quantum well (Ando, Fowler and Stern, 1982), where all the effects of the crystal lattice can be treated within the effective mass approximation (Ashcroft and Mermin, 1976). This picture is valid if the Fermi wavelength of the electron is much longer then the lattice spacing, and the quantum mechanical dynamics of the Bloch electron can be reduced to that of a free itinerant particle with a band mass m_b . In such situations, the only potential energy in the problem corresponds to the Coulomb repulsion E_C between the electrons, which is the dominant energy scale in low carrier density systems. At the lowest densities, $E_C \gg E_F$, and the electrons form a Wigner crystal lattice (Wigner, 1934) to minimize the Coulomb repulsion.

Here, each electron is confined not by an ionic potential, but due to the formation of a deep potential well produced by repulsion from other electrons. The same mechanism prevents double occupation of such localized orbitals, and each electron in the Wigner lattice reduces to a localizes S = 1/2 localized magnetic moment. A Wigner crystal is therefore nothing but a magnetic insulator: a Mott insulator in disguise. At higher densities, the Fermi energy becomes sufficiently large to overcome the Coulomb repulsion, and the Wigner lattice melts (Tanatar and Ceperley, 1989). The electrons then form a Fermi liquid. The quantum melting of a Wigner crystal is therefore a metal-insulator transition, perhaps in many ways similar to a conventional Mott transition. What kind of phase transition is this? Despite years of effort, this important question is still not fully resolved.

What degrees of freedom play the leading role in destabilizing the Wigner crystal as



Fig. 1.9 In a Wigner crystal, each electron is confined to a potential well produced by Coulomb repulsion from neighboring electrons, forming a spin 1/2 local moment. The lowest energy particle hole excitation creates a vacancy-interstitial pair (Cândido, Phillips and Ceperley, 2001), which costs an energy E_{gap} comparable to the Coulomb repulsion.

it melts? Even in absence of an accepted and detailed theoretical picture describing this transition, we may immediately identify two possible classes of elementary excitations which potentially contribute to melting, as follows.

- Collective charge excitations ("elastic" deformations) of the Wigner crystal. In the quantum limit, these excitations have a bosonic character, but they persist and play an important role even in the semi-classical $(k_B T \gg E_F)$ limit, where they contribute to the thermal melting of the Wigner lattice (Thouless, 1978). These excitations clearly dominate in high magnetic field (Chen, Sambandamurthy, Wang, Lewis, Engel, Tsui, Ye, Pfeiffer and West, 2006), where both the spin degrees of freedom and the kinetic energy are suppressed due to Landau quantization.
- Particle-hole excitations leading to vacancy-interstitial pair formation (Fig. 1.9). These excitations have a fermionic character, where the spin degrees of freedom play an important role. Virtual excitations of this type give rise to superexchange processes which produce magnetic correlations (Cândido, Bernu and Ceperley, 2004) within the Wigner crystal. These excitations would exist even if dynamic deformations of the Wigner lattice are suppressed, for example by impurity pinning. Recent quantum Monte-Carlo simulations indicate(Cândido, Phillips and Ceperley, 2001) that the effective gap for vacancy-interstitial pair formation seems to collapse precisely around the melting of the Wigner crystal. If these excitation dominate, then the melting of the Wigner crystal is a process very similar to the Mott metal-insulator transition, and may be expected to produce a strongly correlated $(m^*/m \gg 1)$ Fermi liquid on the metallic side. Behavior consistent with this possibility has recently been documented (Kravchenko and Sarachik, 2004) in several two-dimensional electron systems. A microscopic theory for a simplified model of such Wigner-Mott transitions has recently been solved (Camjayi, Haule, Dobrosavljevic and Kotliar, 2008; Amaricci, Camjayi, Haule, Kotliar, Tanasković and Dobrosavljević, 2010), clarifying the mechanism for the effective mass enhancement on the conducting side.

We should mention, however, that the Wigner crystal melting in zero magnetic field is believed (Tanatar and Ceperley, 1989) to be a weakly first order phase transition. Conventional (e.g. liquid-gas or liquid-crystal) first order transitions are normally associated with a density discontinuity and global phase separation within the coexistence dome. For charged systems, however, global phase separation is precluded by charge

neutrality (Gor'kov and Sokol, 1987). In this case, one may expect the emergence of various modulated intermediate phases, leading to bubble or stripe (Jamei, Kivelson and Spivak, 2005), or possibly even "stripe glass" (Schmalian and Wolynes, 2000) order. While convincing evidence for the relevance of such "nano-scale phase separation" has been identified (Terletska and Dobrosavljević, 2011) in certain systems (Jaroszyński, Andrearczyk, Karczewski, Wróbel, Wojtowicz, Popović and Dietl, 2007), recent work seems to indicate (Waintal, 2006; Clark, Casula and Ceperley, 2009) that such effects may be negligibly small for Wigner crystal melting.

1.2.3 Localization by disorder

A small concentration of impurities or defects simply produces random scattering of mobile electrons. In ordinary metals, the kinetic energy of electrons is generally so large, that the random potential due to impurities can be treated as a small perturbation. In this case, the Drude theory (Ashcroft and Mermin, 1976) applies, where the conductivity takes the form

$$\sigma \approx \sigma_o = \frac{n e^2 \tau_{tr}}{m},\tag{1.17}$$

where n is the carrier concentration, e the electron charge and m its band mass. According to Matthiessen's rule (Ashcroft and Mermin, 1976), the transport scattering rate takes additive contributions from different scattering channels, viz.

$$\tau_{tr}^{-1} = \tau_{el}^{-1} + \tau_{ee}^{-1}(T) + \tau_{ep}^{-1}(T) + \cdots .$$
(1.18)

Here, τ_{el}^{-1} is the elastic scattering rate (describing impurity scattering), and $\tau_{ee}^{-1}(T)$, $\tau_{ep}^{-1}(T)$,..., describe inelastic scattering processes from electrons, phonons, etc. It is important to note that in this picture the resistivity $\rho = \sigma^{-1}$ is strictly a monotonically increasing function of temperature

$$\rho(T) \approx \rho_o + AT^n, \tag{1.19}$$

where A > 0, and the exponent *n* depends on the scattering process (n = 1 for electronphonon scattering; n = 2 for electron-electron scattering, etc.). The residual resistivity $\rho_o = \sigma^{-1}(T = 0)$ is thus viewed as a measure of impurity (elastic) scattering.

In contrast, in low carrier density systems, the impurity potential is comparable or larger than the Fermi energy, and the electrons can get trapped, i.e. "localized" by the impurities. Of course, this process generally leads to a sharp metal-insulator transition only at T = 0, since at finite temperature the electrons can overcome the impurity binding potential through thermal activation. In the low temperature limit, a continuous metal – insulator transition is typically found (Rosenbaum, Andres, Thomas and Bhatt, 1980), where the conductivity decreases in a power – law fashion

$$\sigma(T=0) \sim (n-n_c)^{\mu},$$

where the conductivity exponent μ characterizes the critical point. The precise value of the critical exponent μ , as well as other aspects of the transition depend strongly on the specific form of disorder, and especially on the characteristic lengthscale for disorder. Percolation transition. The simplest situation where disorder can cause electron localization is found in cases where the random potential is sufficiently "smooth". More precisely, consider the situation where the spatial correlation length for the random potential is much larger than the phase coherence length L_{ϕ} (Lee and Ramakrishnan, 1985).



Fig. 1.10 Critical behavior at the bond percolation for a 3D cubic lattice, following (Kirkpatrick, 1973). Percolation probability(dashed line) P(p) and conductance (symbols) G(p)are shown as a function of the bond probability p. The solid line is the prediction of an (approximate) effective medium theory.

In this regime, the influence of the random potential can be described in a semiclassical picture, where quantum interference processes can largely be ignored, and the calculation of the resistivity reduces to solving a percolation problem (Stauffer and Aharoni, 1994) describing a random resistor network.

As the electrons are added to the system, the electron liquid will first fill the deepest potential wells, thus forming small metallic "puddles". At low temperature, regions where the random potential $V(\mathbf{r}) > E_F$ is essentially free of electrons and therefore represent insulating areas. When the electron density (and thus E_F) increases, the metallic puddles grow and eventually connect at the percolation threshold; the system becomes metallic.

The critical behavior of the conductivity within such a percolation approach has been studied in detail (Stauffer and Aharoni, 1994), giving

$$\mu \approx \begin{cases} 2.0 \text{ for } d = 3, \\ 1.3 \text{ for } d = 2. \end{cases}$$
(1.20)

Note (see Fig. 1.10) that such dimensionality-dependent critical behavior is seen only within a narrow critical region. In d = 3, for example, the conductance behavior is rougly linear, in agreement with the prediction of the effective-medium theory (Kirkpatrick, 1973) which does not capture such dimensionality dependence. This theory, as any mean-field description, works equally well in any dimension - everywhere except within a narrow critical region, where long-wavelength fluctuations produce dimensinality-dependent behavior.

The percolation theory predictions can be directly compared experiments, and agreement is found in a number of systems such as granular metals (Beloborodov, Lopatin, Vinokur and Efetov, 2007), where the characteristic inhomogeneities scale is sufficiently large. The percolation scenario seems also to apply (Shahar, Tsui, Shayegan, Shimshoni and Sondhi, 1997) to certain experiments showing apparent violation of the expected quantum critical scaling in quantum Hall plateau transitions. Similar behavior is found, for example, in manganese oxide materials showing colossal magnetoresistance, where disorder induces nano-scale phase separation (Dagotto, 2002), but the transport behavior can be well described using an effective random resistor model and the underlying percolation processes. Here the droplets sizes are not necessarily very large, but the temperatures are sufficiently elevated to produces sufficiently short L_{ϕ} . Such percolative phase coexistence has very recently been observed by nano-scale x-ray imaging on a thermally-driven Mott transition in VO_2 (Qazilbash, Tripathi, Schafgans, Kim, Kim, Cai, Holt, Maser, Keilmann, Shpyrko and Basov, 2011).

In other systems, most notable uncompensated doped semiconductors, low temperature studies find $\mu \approx 0.5$, indicating drastic departure from the percolation picture. Can a better description of quantum effects provide the answer? Or does one have to include strong correlation effects as well? To address this question, it is useful to first discuss the prediction of a full quantum theory of localization for noninteracting electrons in presence of disorder.

Tunneling vs. localization. The semiclassical percolation picture assumes that in the insulating phase the electrons are confined to potential wells. It ignores the possibility of tunneling through barriers separating the wells - a process that is allowed by quantum mechanics. Although the tunneling probability is exponentially small with the barrier size, it always remains finite. Naively, one would expect that the tunneling processes would lead to an exponentially small but finite conductivity even in the regime where classical percolation would predict insulating behavior. According to this argument, quantum mechanical tunneling would smear the metal insulator transition, and no true insulating behavior would be possible, even at T = 0. This argument is, of course, incorrect because is based on an incomplete description of quantum mechanics. It does take into account the tunneling effect, but it ignores the crucial interference processes without which electronic bound states could not be formed. To fully appreciate this, we should recall that interference processes are what gives rise to the formation of quantized electronic orbits even for simple bound states within atoms or molecules.

Basic mechanisms of metal-insulator transitions 21

Anderson localization. The possibility that true electronic bound states can be formed in presence of a random potential was first discussed by Anderson in 1958 (Anderson, 1958). This pioneering work argued that sufficiently strong randomness will localize all the electronic states within a given band, leading to a sharp metal-insulator the transition at T = 0. Anderson's original argument takes the local point of view, which provides a very transparent physical picture of how localization can occur, as follows.

Suppose that an experiment or a computer simulation can examine only local quantities associated with a particular lattice site. Can such a study determine whether the material is a metal or an Anderson insulator? The answer (Anderson, 1958) is somewhat surprisingly - yes! One simply has to determine the escape rate \hbar/τ_{esc} from the local orbital. The recipe how to do this has a long history, and is provided by Fermi's Golden Rule:

$$\frac{\hbar}{\tau_{esc}} = t^2 \rho_{loc}(\varepsilon). \tag{1.21}$$

where t is the appropriate matrix element between the local orbital and its environment. The local density of available states $\rho_{loc}(\varepsilon)$, to which an electron of energy ε can escape, is proportional to the wavefunction amplitude on this site

$$\rho_{loc}(\varepsilon) \sim |\psi_i(\varepsilon)|^2.$$
(1.22)

If electronic states in the relevant energy range are all localized (i.e. bound), then only a small number of such states have appreciable overlap with the given site. Therefore, in the case of the Anderson insulator, the local density of states (LDOS)will consist of only a few discrete δ -function peaks (Fig. 1.11) with appreciable weight - which are very improbable to reside at the Fermi energy.

At strong disorder $W \gg B$, the typical density of available states and thus the *typical escape rate* from a given site can be shown to vanish (Anderson, 1958), and the electron remains localized. In the Anderson insulator, the spectrum of the local environment of any given site has a gap with a certain typical size measuring the localization strength. According to this point of view, the ultimate localization mechanism in both the Mott and the Anderson insulator is somewhat similar: the electron cannot find levels to which it can escape.

Anderson's original arguments demonstrated that sufficiently strong disorder is able to localize all electronic states within a narrow band of electronic states. At weaker disorder, only the states near the band edge are expected to localize, but other states at energies $E > E_c$ - the so-called "mobility edge" - remain extended. As the Fermi energy is increased (for example by carrier doping), the system undergoes an Anderson metalinsulator transition. Following the development of the scaling theories of localization (Abrahams, Anderson, Licciardello and Ramakrishnan, 1979), and especially due to recent progress in numerical studies of the problem, the corresponding critical behavior is now well understood for noninteracting electrons at T = 0, all the electronic states remain localized for dimensions $d \leq 2$, while a continuous metal-insulator transition is found in higher dimensions. According to most recent estimates, the corresponding critical exponent



Fig. 1.11 Wavefunction amplitude $|\psi_i|^2$ of strongly localized states. Since the concept of Anderson localization applies to any wave in random media, recent efforts have documented localization of light in disordered optical media. Shown here are spectra experimentally observed in photonic crystals (Sapienza, Thyrrestrup, Stobbe, Garcia, Smolka and Lodahl, 2010).

in d = 3, and becomes even larger in higher dimensions (see below).

It is interesting to note that the conductivity exponent at d = 3 Anderson transition is not too different from that of percolation theory ($\mu_{perc} \approx 2$). In most experimental systems where the metal-insulator transition has been studied, the observed exponent is much smaller ($\mu \leq 1$). This indicates that disorder-driven mechanisms which ignore electron-electron interactions cannot hope to explain the behavior at the metal-insulator transition.

1.2.4 Basic interaction effects in disordered systems

In most realistic systems both the disordered strength and the electron-electron interactions have comparable magnitudes. In such cases, the localization mechanisms of Mott and Aderson cannot be considered separately, since each will influence and affect the other. Despite recent advances (see Chap.6), a complete theory of such a Mott-Anderson transition remains incomplete. Nevertheless, early physical arguments of Mott (Mott, 1990) already indicate how new and more complicated behavior must emerge when both mechanisms are at play.

The Mott-Anderson transition. To see this most simply, consider a disordered Hubbard model in the strongly localized (atomic) limit. In the absence of disorder, each site has two energy levels, $\varepsilon_0 = 0$ and $\varepsilon_1 = U$, where U is the on-site interaction potential. If the system is half-filled, then each site is singly occupied; the levels ε_1 remain empty. We have one local magnetic moment at each site, and a gap equal to U to charge excitations.

When disorder is added, each of these energy levels is shifted by a randomly fluctuating site energy $-W/2 < \varepsilon_i < W/2$. The situation remains unchanged for W < U, as all the levels $\varepsilon'_1(i) = U + \varepsilon_i$ remain empty (for half-filling the chemical potential is $\mu = E_F = U/2$). For larger disorder, those sites with $\varepsilon_i > U/2$ have the level $\varepsilon'_0(i) = 0 + \varepsilon_i > \mu$ and are empty. Similarly, those sites with $\varepsilon_i < -U/2$ have the excited level $\varepsilon'_1(i) = U + \varepsilon_i < \mu$ and are doubly occupied. Thus for W > U a fraction of the sites are either doubly occupied or empty. The Mott gap is now closed, although a fraction of the sites still remain as localized magnetic moments. We can describe this state as an inhomogeneous mixture of a Mott and an Anderson insulator (as shown in the center of Fig. 12). However, the empty and doubly occupied sites have succeeded to completely fill the gap in the average single particle density-of-states (DOS).



Fig. 1.12 Energy level occupation in the strongly localized (atomic) limit for an Anderson (left), a Mott (right), and a Mott-Anderson (center) insulator. In a Mott-Anderson insulator, the disorder strength W is comparable to the Coulomb repulsion U, and a two-fluid behavior emerges. Here, a fraction of localized states are doubly occupied or empty as in an Anderson insulator. Coexisting with those, other states remain singly occupied forming local magnetic moments, as in a Mott insulator. Note that the spins of the local moments may be randomly oriented indicating the absence of magnetic ordering. The chemical potential is represented by the dashed line.

This physical picture of Mott, which is schematically represented in Fig. 1.12, is very transparent and intuitive. The nontrivial question is how the strongly localized (atomic) limit is approached as one crosses the metal-insulator transition from the metallic side. To address this question one needs a more detailed theory for the metalinsulator transition region, which was not available when the questions posed by Mott and Anderson were put forward.

Coulomb gap. Local moment formation leading to the Mott or the Mott-Anderson insulating state is most important for narrow bands where the on-site Coulomb repulsion ("Hubbard U") dominates. This mechanism is most effective close to half-filling, since

local moment formation requires exactly one electron per orbital. In this regime the long-range (inter-site) component of the Coulomb interaction plays a secondary role, because on-site repulsion opposes charge rearrangement. Such a situation is found in narrow impurity bands (deeply insulating regime) of uncompensated doped semiconductors such as Si:P. Deep in the insulating regime, each electron forms a hydrogenic bound state with exactly one Phosphorus ion, forming a spin S = 1/2 local magnetic moment, and charge rearrangements are suppressed.



Fig. 1.13 Tunneling density of states spectra observed (Lee, Massey, Nguyen and Shklovskii, 1999) accross the metal-insulator transition in Si:B. The Coulomb gap is seen to gradually close and change shape as the transition is crossed. This crossover behavior was interpreted to reflect the emergence of screening as one approaches the metallic phase.

A more complicated situation is found (Shklovskii and Efros, 1984) away from half filling, which can be realized, for example, in partially compensated Si:P,B. Here the electrons can occupy different localized states, and many charge rearrangements are possible. This is the regime considered by the well-known theory of Efros and Shklovskii (Efros and Shklovskii, 1975; Shklovskii and Efros, 1984), which focuses on a classical model of spinless electrons distributed among stongly localized states, as given by the Hamiltonian

$$H = \sum_{j \neq i} \frac{e^2}{\kappa |\mathbf{r_i} - \mathbf{r_j}|} (n_j - \overline{n}) (n_j - \overline{n}) + \sum_{j \neq i} \varepsilon_i n_i.$$
(1.24)

Here $n_j = 0, 1$ is the occupation number of the remaining localized states with (bare) energy ε_i at position \mathbf{r}_i , \overline{n} is the average occupation per site, and κ is the dielectric constant of the insulator. For localized electrons, the single-particle (tunneling) density of electronic states (DOS) is then simply the probability distribution of the local energy levels

$$N(\varepsilon) = <\delta(\varepsilon - \varepsilon_i) > \tag{1.25}$$

of an electron occupying a localized state is shifted (renormalized) by the electrostaic potential produced by the electrons on all remaining sites.

$$\varepsilon_i \to \varepsilon_i^R = \varepsilon_i + \sum_{j \neq i} \frac{e^2}{\kappa |\mathbf{r_i} - \mathbf{r_j}|} (n_j - n).$$
 (1.26)

This electrostatic shift depends, of course, on the precise electronic configuration, favoring those charge configurations which lower the energies of the occupied states. In absence of disorder, this effect leads to charge ordering - Wigner crystallization - opening a hard gap at the Fermi energy.

When sufficiently strong disorder is present, the Wigner gap is smeared, leading to the soft "Coulomb gap" (Fig. 1.13). The original argument of Efros and Shklovskii (Efros and Shklovskii, 1975) rested on a stability argument for the ground state with respect to any single-particle displacement from a given occupied site i to an empty site j. Stability requires every such excitation to cost positive energy, giving

$$\varepsilon_j - \varepsilon_i - \frac{e^2}{\kappa |\mathbf{r_i} - \mathbf{r_j}|} > 0.$$
 (1.27)

If the states i and j are within energy ε from the Fermi level, i.e. $|\varepsilon_i - \varepsilon_F| < \varepsilon$, then their typical separation in space is $r(\varepsilon) = e^2/\kappa\varepsilon$ is large if ε is small. The DOS $N(\varepsilon) \sim \frac{d}{d\varepsilon}[r^{-3}(\varepsilon)]$ then has a soft gap around the Fermi energy. In three dimensions the result is

$$N(\varepsilon) = \frac{3\kappa^3}{\pi e^6} (\varepsilon - \varepsilon_F)^2.$$
(1.28)

More generally, for interactions of the form $V(r) \sim r^{-\alpha}$, with $\alpha < d$ (in *d* spatial dimensions), a generalization of this argument gives (Pankov and Dobrosavljević, 2005) $N(\varepsilon) \sim (\varepsilon - \varepsilon_F)^{\beta}$, with $\beta = (d - \alpha)/\alpha$. Arguing that incomplete (scale-dependent) screening modifies the form Coulomb interaction within the quantum critical region of the metal-insulator transition, recent work suggested (Lee, Massey, Nguyen and Shklovskii, 1999) that in this case $\alpha = 2$, leading to $\beta = 1/2$ in d = 3. While such critical scaling seems consistent with the experimental results on Si : B, a more

precise treatment of quantum effects will be needed for a more convincing theory of the Coulomb gap in the quantum critical region.

It is very important to reiterate that the Anderson localization mechanism by itself does not lead to opening of any kind of gap at the transition. This scenario, which provided a popular and attractive scenario for interpreting many transport experiments, rests on the concept of a "mobility edge" - the energy separating the extended from the localized states. The critical concentration then obtains when all the localized states are filled up and the Fermi energy reaches the mobility edge. Precisely at the critical point one expects (Mott, 1990) the electronic states "above" the Fermi energy to be extended, while those "below" the mobility edge to remain localized. A test of these ideas has very recently become possible through high resolution scanning tunneling microscopy (STM) experiments (Richardella, Roushan, Mack, Zhou, Huse, Awschalom and Yazdani, 2010) which can directly determine the degree of wavefunction localization in an energy-resolved fashion. In contrast to the conventional mobility edge scenario, these experiments provided striking evidence that, close to the critical concentration, the electronic states precisely at the Fermi energy are the ones most strongly localized. This experiment, which will be discussed in more detail in Chapter 7, provided direct evidence that interaction effects cannot be neglected near the metal-insulator transition, where pseudogap opening plays a key role in controlling the localization of electronic states. Is the relevant interaction mechanism directly related to Coulomb glass phenomena? Only time will tell. Still, all these experiments make it clear that the fundamental – but yet unresolved – physics questions posed by the early work of Mott cannot be ignored.

Coulomb glass. The long-range Coulomb interactions produce, however, another important effect. Because Coulomb repulsion favors a uniform charge configuration while disorder opposes it, these competing interactions give rise to frustration and the emergence of many meta-stable electronic states. As described in Chapter 8, this typically leads gradual glassy freezing of electrons, to slow relaxation and "aging", in a fashion surprisingly similar to glassy phenomena in spin glasses or super-cooled liquids.

The precise relation of the Coulomb gap formation and the glassy freezing has, however, long remained controversial and ill-understood (Grannan and Yu, 1993). On the one hand, even the early arguments of Efros and Shklovskii indicate that the long-range nature of the Coulomb interaction is a key feature for the formation of the Coulomb gap. On the other, thermal and/or quantum fluctuations can allow charge rearrangements, generically leading to phenomenon of screening, practically eliminating the long-range part of the interaction. In contrast, if the electrons are (partially or fully) frozen in a glassy state, then screening may remain incomplete, allowing the long-range nature of the Coulomb interaction to manifest itself. This physically plausible idea (Lee and Ramakrishnan, 1985) has found support in very recent theoretical work (Pastor and Dobrosavljević, 1999; Pastor, Dobrosavljević and Horbach, 2001; Pankov and Dobrosavljević, 2005), which argues that the two phenomena typically go hand-in-hand.

Both the formation of the Coulomb gap and the emergence of glassy features are, at this time, well established features in strongly disordered insulators. But how should this influence the approach to the metal-insulator transition? Recent theoretical (Pastor and Dobrosavljević, 1999; Dobrosavljević, Tanasković and Pastor, 2003*b*) and experimental (Bogdanovich and Popović, 2002; Jaroszynski, Popovic and Klapwijk, 2004) works have suggested that in some cases it may dramatically affect the critical region, perhaps even leading to an intermediate metallic-glass phase with unusual transport properties (Dalidovich and Dobrosavljević, 2002). This important question remains far from settled. Still, its fundamental importance has very early been recognized by Mott (Mott, 1990), who noted that the phenomenon of screening must be dramatically modified as one crosses from a metal to an insulator. In short, the localization of electrons immediately produces the demise of screening, so the two phenomena must be profoundly linked. Mott's dream was to understand how this "unscreening" occurs at the metal-insulator transition, a physical question of basic importance, but one that is typically not addressed by most conventional theories, thus remaining a major challenge for future work.

1.3 Current theories of the metal-insulator transition

1.3.1 MIT as a critical point

Absence of minimum metallic conductivity. The early ideas of Mott and Anderson identified the basic mechanism for the metal-insulator transition, but did not provide specific and detailed prediction for the critical behavior, or even the precise nature of this phase transition. In fact, up to the late 1970s, Mott's arguments (Mott, 1990) suggested that the transition is discontinuous, where a minimum metallic conductivity should exist on the metallic side even at T = 0. Mott's early argument examined the transport behavior based on Drude's picture, where increasing disorder simply reduces the elastic mean-free path ℓ . Since the scattering rate τ_{tr}^{-1} from any impurity assumes a maximum possible value which is finite (the so-called "unitarity limit"), the corresponding mean-free path $\ell = v_F \tau_{tr}$ cannot be shorter then a microscopic lower cutoff a of the order of the lattice spacing. Therefore, it was argued, the conductivity of any metal is bounded from below by the "Mott limit"

$$\sigma \ge \sigma_{\min} = \frac{ne^2a}{mv_F}.$$

Early low temperature experiments on many materials seemed to confirm these predictions by only reporting metallic conductivities in excess of σ_{\min} . The metal-insulator transition in disordered systems was thus assumed to have a first-order character, similarly as the Mott transition in clean systems. Similarly, high temperature behavior in a number of metallic systems was found to display "resistivity saturation", where the Mott limit is approached due to incoherent (inelastic) scattering.

With the development of more advance cryogenic techniques, lower temperatures and more precise measurements became available. Our perspective on the fundamental nature of the transition has been deeply influenced by the ground-breaking experiments on Si:P in early 1980s (Rosenbaum, Andres, Thomas and Bhatt, 1980; Paalanen, Rosenbaum, Thomas and Bhatt, 1982). These experiments provided evidence (Fig. 1.14) of metallic conductivities as much as two orders of magnitude smaller then σ_{min} . They also made it clear that the metal-insulator transition in doped semiconductors

is a continuous (second order) phase transition (Paalanen, Rosenbaum, Thomas and Bhatt, 1982), which bears many similarities to conventional critical phenomena. This important observation has sparked a veritable avalanche of experimental (M. A. Paalanen, 1991; Sarachik, 1995) and theoretical (Wegner, 1976; Wegner, 1979; Abrahams, Anderson, Licciardello and Ramakrishnan, 1979; Schaffer and Wegner, 1980) works, most of which have borrowed ideas from studies of second order phase transitions. Indeed, many experimental results were interpreted using scaling concepts (Lee and Ramakrishnan, 1985), culminating with the famed scaling theory of localization (Abrahams, Anderson, Licciardello and Ramakrishnan, 1979), and the subsequent extensions to incorporate the interaction effects (Finkel'stein, 1983; Finkel'stein, 1984; Castellani, Castro, Lee and Ma, 1984; Belitz and Kirkpatrick, 1994).



Fig. 1.14 Critical behavior of the conductivity extrapolated to $T \to 0$ for uncompensated Si:P (Rosenbaum, Andres, Thomas and Bhatt, 1980). Sharp power-law behavior with critical exponent $\mu \approx 1/2$ is extending over a surprisingly large concentration range. Finite values of the conductivity much smaller then σ_M (shown by arrow) are observed close to the transition.

Phenomenological scaling formulation. Microscopic theories describing the MIT remain controversial and somewhat incomplete. We should stress, however, that scaling behavior near second order phase transitions is a much more robust and general property then any particular approximation scheme or microscopic model. Historically, the scaling hypothesis of Widom (Widom, 1965) has been put forward for conventional (classical) critical phenomena much before the microscopic theory of Wilson (Wilson and Kogut, 1975) became available. It has provided crucial guidance for experimentalist to systematically analyze the experimental data, and has provided a framework and direction for the development of microscopic theories.

A phenomenological scaling hypothesis can be formulated for quantum criticality (Sachdev, 2011) as well, in direct analogy to conventional critical phenomena. In particular, if the scaling description is valid, then a single correlation length $\xi \sim \delta n^{-\nu}$ exists characterizing the system, and the corresponding time scale $\tau_{\xi} \sim \xi^{z}$, both of which diverge in a powerlaw fashion at the critical point. Here, $\delta n = (n - n_c)/n_c$ is the dimensionless distance from the transition, and we have introduced the correlation length exponent ν and the "dynamical exponent" z. Because of the Heisenberg Uncertainty Principle, the corresponding energy (temperature) scale

$$T^* \sim \frac{\hbar}{\tau_{\xi}} \sim \delta n^{\nu z}$$

vanishes as the critical point is approached.

For the MIT the conductivity plays a role similar to an order parameter, as it vanishes at $T \rightarrow 0$ in the insulating phase. Its sharp critical behavior at T = 0 is rounded at finite temperature, suggesting a scaling behavior similar to that of a ferromagnet in an external (symmetry breaking) field. The conductivity can therefore be written in a scale invariant form as

$$\sigma(\delta n, T) = b^{-\mu/\nu} f_{\sigma} (b^{1/\nu} \delta n, b^z T).$$

Here, T is the temperature, b is the length rescaling factor, and μ is the conductivity exponent.

The T = 0 behavior $\sigma (T = 0) \sim \delta n^{\mu}$ can be obtained by working at low temperatures and choosing $b = \delta n^{-\nu} \sim \xi$. We obtain the scaling form

$$\sigma(T) = \delta n^{\mu} \phi_{\sigma}(T/\delta n^{\nu z}), \qquad (1.29)$$

where $\phi_{\sigma}(y) = f_{\sigma}(1, y)$. Finite temperature corrections in the metallic phase are obtained by expanding

$$\phi_{\sigma}(y) \approx 1 + ay^{\alpha},\tag{1.30}$$

giving the low temperature conductivity of the form

$$\sigma(\delta n, T) \approx \sigma_o(\delta n) + m_\sigma(\delta n) T^\alpha. \tag{1.31}$$

Here, $\sigma_o(\delta n) \sim \delta n^{\mu}$, and $m_{\sigma}(\delta n) \sim \delta n^{\mu - \alpha \nu z}$. Since the form of the scaling function $\phi_{\sigma}(y)$ is independent of the distance to the transition δn , the exponent α must take an universal value in the entire metallic phase, and therefore can be calculated by

perturbation theory at weak disorder. For example, the interaction corrections in d = 3 (Lee and Ramakrishnan, 1985) lead to $\alpha = 1/2$.

This scaling argument provides a formal justification for using the predictions from perturbative quantum corrections as giving the leading low temperature dependence in the entire metallic phase. Note, however, that the prefactor $m_{\sigma}(\delta n)$ (i.e. its dependence on δn) is not correctly predicted by perturbative calculations, since it undergoes Fermi liquid renormalizations which can acquire a singular form in the critical region near the metal-insulator transition.

The temperature dependence at the critical point (in the critical region) can be obtained if we put $\delta n = 0$, and choose $b = T^{-1/z}$, giving

$$\sigma_c(T) = \sigma(\delta n = 0, T) \sim T^{\mu/\nu z}.$$
(1.32)

We can also write

$$\sigma(n,T) = T^{\mu/\nu z} \phi_{\sigma}(T/T_o(n)), \qquad (1.33)$$

where $\phi_{\sigma}(x) = x^{\mu/\nu z} \widetilde{\phi}_{\sigma}(x)$, and the crossover temperature $T_o(\delta n) \sim \delta n^{\nu z}$.

How to experimentally find quantum criticality?. To demonstrate quantum critical scaling around a MIT one must adopts the following systematic procedure in analyzing experimental data:

- 1. Plot $\sigma(\delta n, T)$ vs. T for several carrier concentrations n. Simple powerlaw behavior $\sigma_c \sim T^x$ is expected only at the critical point, and we can identify the critical concentration $(n = n_c)$ as the only curve that looks like a straight line when the date are plotted on a log-log scale.
- 2. From the slope of the $\sigma_c(T)$ we find the critical exponent $x = \mu/\nu z$.
- 3. Having determined $\sigma_c(T) = \sigma(0,T)$ we can now plot $\phi_\sigma(T/T_o(\delta n)) = \sigma(\delta n,T)/\sigma_c(T)$ as a function of $T/T_o(\delta n)$. The crossover temperature is determined for each concentration n in order to collapse all the curves on two branches (metallic and insulating) of the scaling function $\phi_\sigma(y)$. This procedure does not assume any particular functional form (density dependence) for the crossover scale $T_o(\delta n)$.
- 4. Next, we plot $T_o(\delta n)$ as a function of δn on log-log scale to determine the corresponding exponent νz . If scaling works, then $T_o(\delta n)$ should vanish as the transition is approached, and we expect to find the same exponent from both sides. The conductivity exponent is then obtained from $\mu = x\nu z$.
- 5. A crosscheck can be obtained from extrapolating the metallic curves $\sigma(\delta n, T) \longrightarrow \sigma_o(\delta n)$ to $T \longrightarrow 0$, and by determining the exponent μ from the relation $\sigma_o(n) \sim \delta n^{\mu}$.

All the above expressions are quite general, and can be considered to be a phenomenological description of the MIT. This is of particular importance in instances where a scaling approach is utilized to systematically analyze the experimental data in instances where an accepted microscopic theory is not available. Such a situation is found in several two-dimensional systems, where beautiful and convincing scaling behavior is observed, providing evidence that the MIT is a well defined quantum critical point. How to theoretically approach quantum criticality?. To better understand the *physical content* of the scaling approach to the MIT, we should contrast it to standard approaches to critical phenomena, which are by now perfectly well understood. In most cases, the theory is built based on the emergence of spontaneous symmetry breaking within the low temperature (ordered) phase. Based on identifying the appropriate order parameter φ describing such symmetry breaking, one typically proceeds in the following steps:

- 1. Formulate an appropriate Landau theory, which defines how the free energy $F[\varphi]$ depends on the (spatially fluctuating) order parameter $\varphi(\mathbf{x})$.
- 2. Mean-field theory (MFT) is obtained by minimizing $F[\varphi]$ and ignoring the spatial fluctuations of $\varphi(\mathbf{x})$.
- 3. Examine the effects of long wavelength spatial fluctuations of the order parameter beyond MFT.
- 4. The most singular effects of spatial fluctuations are found close to critical points, and are re-summed using renormalization group (RG) methods.

When this program is implemented in practice, one finds that the mean-field description (steps 1 and 2) suffices everywhere except in a very narrow interval around the critical point. In fact, it is precisely by examining the leading corrections to MFT one is able to theoretically estimate the size of the so-called "Ginzburg" region where non-MFT behavior can be observed. Accounting for non-MFT behavior within such a critical region is much more difficult, and requires the powerful arsenal of renormalization group (RG) methods (Goldenfeld, 1992). The practical calculations simplify considerably near the upper critical dimension d_{uc} , where the fluctuations corrections are logarithmically weak, and can be effectively re-summed by a perturbative RG methods, using $\varepsilon = d_{uc} - d$ as a small parameter in the theory (Wilson and Fisher, 1972). For standard magnetic critical phenomena $d_{uc} = 4$, this RG program has been effectively implemented, and all the appropriate critical exponents calculated to leading order in $\varepsilon = 4 - d$. When the results are extrapolated to d = 3, very impressive agreement with both experiments and numerical simulation results has obtained. The theory of conventional (thermal) critical phenomena can therefore be considered a closed book.

What to do if an order-parameter description is not available?. We emphasize that the above "standard" approach to criticality relies on being able to identify an appropriate symmetry breaking scheme. For phenomena such as magnetic and charge ordering or superconductivity, this program can be straightforwardly extended to the quantum domain, as first discussed by John Hertz (Hertz, 1976). In such cases, especially for insulating magnets, the familiar approach of Landau theory and weakcoupling RG methods has been studied in detail, and met some success in describing quantum criticality (Sachdev, 2011).

When it comes to the metal-insulator transition, the situation is more complicated. Here, despite convincing experimental evidence for criticality, the conventional approach cannot be applied directly. The fundamental difficulty lies in the absence of an obvious symmetry breaking scheme needed to build a Landau theory. For this reason, a simplistic mean-field description of the MIT is not readily available, and one is forced to look for alternative approaches.

A clue on how this may be possible is again found by analogy to conventional critical phenomena, where one generally expects the fluctuations to increase in importance in low dimensions. Their effects are particularly strong near the lower critical dimension (LCD) d_{lc} , where they are able to completely suppress the ordering. These phenomena allow for a particularly elegant approach in systems with continuous broken symmetry where $d_{lc} = 2$. Here, an alternative perturbative RG treatment based on an $\varepsilon = d - 2$ expansion has been developed following early ideas of Polyakov (Polyakov, 1975).

The simplest example is the behavior of a Heisenberg magnet near d = 2, where one examines the low temperature spin-wave corrections to the spin stiffness. Finite corrections are found for d > 2, while logarithmic singularities of the form $T \ln L$ (T is the temperature and L is the system size) arise precisely in d = 2. This result indicates the instability of the ordered phase due to infinitesimal thermal fluctuations indicating that $d_{lc} = 2$, and allows for a perturbative RG treatment based on expanding around two dimensions (Nelson and Pelcovits, 1977).

In the case of disorder-driven MITs, one should examine the effects of weak disorder on the stability of the metallic phase. If similar singular corrections arise in low dimensions, then one can not only hope to identify the LCD, but should also be able to develop a perturbative RG scheme. This elegant approach does not require developing an appropriate symmetry breaking scheme, thus bypassing the essential stumbling block in the theory for the MIT. However, it focuses on those physical processes that describe how weak disorder modifies a clean Fermi liquid. In those instances where the system is close to instabilities of the clean system, this approach may be insufficient because new types of low energy excitations may become important. Such a situation may be found if the clean system is sufficiently close to Mott or Wigner transitions, in which case strong correlation effects may require a different theoretical framework and approach. Independent of these issues, the physical content of theories describing perturbative disorder effects within conventional Fermi liquids is sufficiently rich and nontrivial, and in the following we discuss its basic ideas and results.

1.3.2 Scaling theories of disorder-driven transitions

Phenomenological β -function. An elegant and compact description of the scaling behavior around a critical point is provided by the β -function formulation. What we want to emphasize here is that the β -function description is simply a alternative language one can use, rather then a microscopic theory. However, its straightforward application to the MIT is based on several implicit assumptions, that allow for a simplified phenomenology, and which we discuss as follows.

In its original formulation as presented by the "gang of four" (Abrahams, Anderson, Licciardello and Ramakrishnan, 1979), the β -function describes how the conductance of the system changes with the (effective) system size. In principle, one could imagine taking a finite size chunk of disordered metal, and attaching it to contacts. If the experiment is repeated for different sample sizes, then one could experimentally determine how the conductance depends on the system size.

Why use the conductance and what do we expect to find? The essential physics is easy to see by thinking about what happens far from the MIT. In a good metal disorder is weak, and the conductivity σ is large and finite even at T = 0, and the standard Drude theory applies. From Ohm's law, the conductance g then scales with the system size L as

$$g_{met}(L) = \sigma L^{d-2}.$$

For increasing system sizes (in d > 2), the conductance grows as L^{d-2} . In the opposite limit of very strong disorder, we expect all the electrons to form bound (localized) states with impurities. If ξ is the characteristic (localization) length of these bound states, then the conductance is expected to decrease exponentially

$$g_{ins}(L) \sim \exp\{-L/\xi\}.$$

More generally, we may expect that g(L) do increase with L in the metal, and decrease in the insulator. What is not a priory obvious is how g(L) behaves around the transition. The seminal work on the scaling theory for Anderson localization concentrated on the logarithmic rate of change of the conducatnce with lengthscale, by defining the " β -function"

$$\beta(g) = \frac{d(\ln g)}{d\ln L}.$$
(1.34)

This quantity is expected to be positive in a metal and negative in the insulator. Its precise form, or a possible dependence on the sample size L, is not a priori clear.

To make more specific predictions on the critical regime, the "gang of four" made two *key assumptions*, as follows:

- 1. $\beta(g)$ is a function on g only, but does not depend explicitly on L.
- 2. $\beta(g)$ is a smooth (analytic) function near the transition.

In particular, in an Ohmic metal, we find

$$\beta_{met} = d - 2, \tag{1.35}$$

while in the localized insulator

$$\beta_{ins} = \ln g. \tag{1.36}$$

Since $\beta_{met} > 0$, and $\beta_{ins} < 0$, assumptions (1) and (2) then suggest that $\beta(g)$ has to change sign at some *finite* value of the conductance $g = g_c$, and we can write

$$\beta(g) \approx s \ln(g/g_c), \tag{1.37}$$

where $s = \beta'(g_c)$ is the critical slope of the β -function. Defining the logarithmic variable $t = \ln(g/g_c)$, we can now integrate the β -function equation

$$\frac{d(\ln t)}{d\ln L} \approx s,$$

from the microscopic cutoff ℓ to the sample size L, and write

$$t(L) = t_o (L/\ell)^s.$$

This integration has to be carried out up to the lengthscale (i.e. the correlation length) $L = \xi$ such that the renormalized distance to the transition $\delta g(L) = (g(L) - g_c)/g_c \approx t(L) \sim O(1)$. Since at short scales $t_o = t(\ell) \sim \delta g_o \sim \delta n$, we find

$$\xi \sim t_o^{1/s} \sim \delta n^{1/s}.$$

Thus, the correlation length exponent

$$\nu = 1/s.$$

At this scale the conductivity is expected to saturate to a (size-independent) macroscopic value

$$\sigma \approx g_c \xi^{d-2} \sim \delta n^{(d-2)\nu}.$$

We conclude that the assumptions implied by the β -function formultion, i.e. the scaling theory of localization predict the validity of "Wegner scaling" for the conductivity exponent

$$u = (d-2)\nu.$$

It is worth emphasizing that postulating a particular form for the β -function is completely equivalent to formulating a scaling hypothesis for the critical behavior, provided that the conductance is assumed to be finite at the transition. Under these conditions, the scaling hypothesis states that the T = 0 conductance of a finite size system takes a scaling form

$$g(\delta n, L) = f_{\sigma} (b^{1/\nu} \delta n, b/L).$$

Choosing $b = \delta n^{-\nu} \sim \xi$, we can write $g(\delta n, L) = \psi(\xi/L)$, where $\psi(x) = f_{\sigma}(1, x)$. The condition that the conductance is finite at the transition is equivalent to require that $\psi(0) = g_c$ is a finite constant. Using the definition of the β -functon, we then find

$$\beta(g) = x^{-1}\psi'(x)/\psi(x),$$

where $x = \psi^{-1}(g)$. The form of the β -function can be directly extracted from the experimental data, provided that an appropriate scaling behavior is found.

Is there a β -function for percolation?. The β -function formulation of the "gang of four" (Abrahams, Anderson, Licciardello and Ramakrishnan, 1979) may be viewed as a convenient phenomenological description of how the conductance depends on lengthscale. While its implicit assumptions prove correct for the problem of Anderson localization of noninteracting electrons, one may ask the same question for other models of the metal-insulator transition. In particular, the percolation problem represents a consistent description of the transition in the semiclassical limit. Since percolation (Stauffer and Aharoni, 1994) is a well characterized (classical) critical phenomenon, powerlaw scaling of all quantities is still valid, and many of the same questions raised by the "gang of four" can be again posed.

Since d = 2 remain above the lower critical dimension for percolation, Wegner scaling cannot be valid in this case. How is thecaling behavior modified in this case, and what would the β -function look like? Can it even be defined? How is the finite-temperature scaling modified, and what form does it take?

These questions are important, since the precise answer allows us to distinguish experimental systems where percolation behavior dominates from those where genuine quantum critical behavior is at play. In the case of the percolation transition, we can give precise and rigorous answers to all these questions, as we discuss in the following.

To discuss transport behavior near the percolation transition, consider a resistor network corresponding to a random mixture of a metallic and and insulating component. Let the conductivities of the respective components be $\sigma_M(T)$ and $\sigma_I(T)$, both of which remain finite at $T \neq 0$, but with $\sigma_M(0) = \sigma_o \neq o$, while $\sigma_I(0) = 0$. As the relative fraction x of the insulating component increases past the percolation concentration x_c , the overall conductivity of the network behaves as

$$\sigma(x, T = 0) \sim (x_c - x)^{\mu},$$
(1.38)

but such a sharp critical behavior emerges only at T = 0. At any finite temperature the sharp percolation transition is smeared, similarly as when a symmetry breaking field is turned on in conventional critical phenomena. The family of conductivity curves generated by varying the percolation concertation x and temperature T at first glance looks very similar to those expected at any quantum localization transition, where sharp critical behavior also emerges only at T = 0.

How can we distinguish the two phenomena? The simples way to do so is by focusing on the behavior at the the critical concetration, where the finite temperature conductivity is determined (Straley, 1977) by the conductivity of the insulating components

$$\sigma(x_c, T) \sim (\sigma_I(T))^u. \tag{1.39}$$

The corresponding critical exponent u = 1/2 in d = 2, and $u \approx 0.7$ in d = 3 (Straley, 1977). Therefore, one may first determine the location of the critical point by extrapolating the conductivity to T = 0 from the metallic side. One should then plot the temperature dependence at the critical point. At any quantum critical point, we generally expect a powelaw temperature dependence, $\sigma_c(T) \sim T^{(d-2)/x}$. In contrast, the conductivity of any insulator typically takes a exponential form

$$\sigma_I(T) \sim \exp\{-(T_o/T)^\alpha\} \tag{1.40}$$

where $\alpha = 1$ for simple activated behavior, $\alpha = 1/(d+1)$ for Mott variable-range hopping (Mott, 1990), or $\alpha = 1/2$ for Efros-Shklovskii hopping (Shklovskii and Efros, 1984). Since the measured conductance is expected to be a power of σ_I , we conclude that in presence of percolation, the conductivity will assume an insulating-like (exponential) temperature dependence on only in the insulating phase, but even at the critical point. Thus, when the conductivity is plotted as a function of temperature on a log-log scale, one will find a single straight line (indicated powerlaw behavior at the critical concentration) in the case of quantum criticality, but no such curve will be found in case of percolation.

One can be even more precise, and specify the precise scaling behavior around such a percolation transition, which can be used to perform an alternative scaling analysis and collapse the experimental curves in presence of percolation. In this case, we expect (Stauffer and Aharoni, 1994) that the conductivity should assume the following scaling form

$$\sigma_{perc}(\delta x, h, L) = b^{-(d-2+\zeta)} f_{perc}(b^{1/\nu} \delta x, h b^{z'}, b/L).$$
(1.41)

Here, $\delta x = (x_c - x)$ is the distance to the critical point, $h = \sigma_I / \sigma_M$ plays the role of the symmetry breaking field. Choosing $b^{1/\nu} \delta x = 1$ and taking $L \to \infty$, we can write

$$\sigma_{perc}(\delta x, h) = \delta x^{\mu} \phi_{perc}(h/\delta x^{\nu z'}).$$
(1.42)

The conductivity exponent

$$\mu = (d - 2 + \zeta)\nu, \tag{1.43}$$

which replaces Wegner scaling of the quantum case. Thus, to collapse all the experimental curves, the argument of the scaling function should contain $h \sim \sigma_I(T)$, and not the temperature T. Working at the critical point ($\delta x = 0$), we similarly find the exponential relation

$$u = \frac{d-2+\zeta}{z'}.\tag{1.44}$$

Finally, let us examine the finite size scaling behavior at h = 0. Working again at the critical point, we find

$$\sigma_c^{perc}(L) \sim L^{-(d-2+\zeta)},\tag{1.45}$$

so that the critical conductance

$$g_c^{perc}(L) \sim L^{-\zeta},\tag{1.46}$$

revealing the physical meaning of the anomalous dimension ζ , which describes transport on the critical percolation cluster. As we can see, the critical conductance vanishes in the case of percolation, in contrast to the scaling theory of localization. We conclude that although percolation does display conventional powerlaw finite size scaling at the critical point, a β -function description is not possible. This behavior reflects the fact that d = 2 is not the lower critical dimension for percolation. The anomalous dimension ζ is very generally expected to vanish at ordinary quantum critical points describing conductor-insulator transitions. This result can be shown (Wen, 1992) to very generally follow from charge conservation for any quantum criticality displaying simple single-parameter scaling behavior. Although more complicated quantum scenarios are in principle possible, no microscopic quantum model has been identified to date showing $\zeta \neq 0$, or equivalently a lower critical dimension $d_{lc} < 2$.

Perturbative quantum corrections in disordered metals. What is the physical mechanism that invalidate's Mott's bound on impurity scattering? To understand this, recall that Mott used Drude's picture, where scattering processes from each impurity or defect are assumed to independent and uncorrelated. This assumption is indeed justified at weak enough disorder, where the Drude prediction is recovered as a leading order contribution. For stronger disorder, multiple-scattering processes cannot be ignored, and they provide the so-called "quantum corrections" to Drude theory. In good metals the magnitude of the quantum corrections is generally small, modifying the conductivity by typically only a fraction of a percent. In this regime, the quantum corrections can be systematically obtained as next-to-leading corrections within weak-disorder perturbation theory for impurity scattering. Detailed calculations and classification of all such perturbative quantum corrections has been carried out in the late 1970s and early 1980s, and are by now well understood (Lee and Ramakrishnan, 1985). They consist of several additive terms,

$$\sigma = \sigma_o + \delta \sigma_{wl} + \delta \sigma_{int},$$

corresponding to the so-called "weak localization" and "interaction" corrections. These "hydrodynamic" corrections are dominated by infrared singularities, i.e., they acquire non-analytic contributions from small momenta or equivalently large distances. Specifically, the weak localization corrections take the form

$$\delta\sigma_{wl} = \frac{e^2}{\pi^d} \left[l^{-(d-2)} - L_{Th}^{-(d-2)} \right], \qquad (1.47)$$

where $l = v_F \tau$ is the mean free path, d is the dimension of the system, and L_{Th} is the length scale over which the wave functions are coherent. This effective system size is generally assumed to be a function of temperature of the form $L_{Th} \sim T^{p/2}$, where the exponent p depends on the dominant source of decoherence through inelastic scattering.

The situation is simpler in the presence of a weak magnetic field or magnetic impurities. Here the weak localization corrections are suppressed and the leading dependence comes from the interaction corrections first discovered by Altshuler and Aronov (Altshuler and Aronov, 1979)

$$\delta\sigma_{int} = \frac{e^2}{\hbar} (c_1 - c_2 \widetilde{F}_{\sigma}) (T\tau)^{(d-2)/2}.$$
(1.48)

Here, c_1 and c_2 are constants, and \widetilde{F}_{σ} is an interaction amplitude.

In d = 3, this leads to a square-root singularity $\delta \sigma_{int} \sim \sqrt{T}$, and to a more singular logarithmic divergence $\delta \sigma_{int} \sim \ln(T\tau)$ in d = 2. These corrections are generally more singular than the temperature dependence of the Drude term, and thus they are easily identified experimentally at the lowest temperatures. Indeed, the $T^{1/2}$ law is commonly observed (Lee and Ramakrishnan, 1985) in transport experiments in many disordered metals at the lowest temperatures, typically below 500 mK.

Similar corrections have been predicted for other physical quantities, such as the tunneling density of states and, more importantly, for thermodynamic response functions. As in Drude theory, these quantities are not expected to be appreciably affected by noninteracting localization processes, but singular contributions are predicted from interaction corrections. In particular, corrections to both the spin susceptibility χ , and the specific heat coefficient $\gamma = C_V/T$ were expected to take the general forms

$$\delta \chi \sim \delta \gamma \sim T^{(d-2)/2}$$

again leading to logarithmic corrections in d = 2.

As in conventional Fermi liquid theories, these corrections emerged already when the interactions were treated at the lowest, Hartree-Fock level, as done in the approach of Altshuler and Aronov (Altshuler and Aronov, 1979). Higher order corrections in the interaction amplitude were first incorporated by Finkelshtein (Finkel'stein, 1983;

Finkel'stein, 1984), demonstrating that the predictions remained essentially unaltered, at least within the regime of weak disorder. In this sense, Fermi liquid theory has been generalized to weakly disordered metals, where its predictions have been confirmed in numerous materials (Lee and Ramakrishnan, 1985).

And erson transition in $2 + \varepsilon$ dimensions. The essential idea of these approaches focuses on the fact that a weak, logarithmic instability of the clean Fermi liquid arises in two dimensions, suggesting that d = 2 corresponds to the lower critical dimension of the problem. In conventional critical phenomena, such logarithmic corrections at the lower critical dimension typically emerge due to long wavelength fluctuations associated with spontaneously broken continuous symmetry. Indeed, early work of Wegner emphasized (Wegner, 1979) the analogy between the localization transition and the critical behavior of Heisenberg magnets. It mapped the problem onto a field theoretical nonlinear σ -model and identified the hydrodynamic modes leading to singular corrections in d = 2. Since the ordered (metallic) phase is only marginally unstable in two dimensions, the critical behavior in d > 2 can be investigated by expanding around two dimensions. Technically, this is facilitated by the fact that in dimension $d = 2 + \varepsilon$ the critical value of disorder W for the metal-insulator transition is very small $(W_c \sim \varepsilon)$, and thus can be accessed using perturbative renormalization group (RG) approaches in direct analogy to the procedures developed for Heisenberg magnets. In this approach (Schaffer and Wegner, 1980; Abrahams, Anderson, Licciardello and Ramakrishnan, 1979), conductance is identified as the fundamental scaling variable associated with the critical point, which is an unstable fixed point of the RG flows. This RG calculation provides a scaling description predicting how the conductance depends on the system size, and thus produces the desired β -function in $d = 2 + \varepsilon$ dimensions. To leading order in ε the resulting critical exponent ν is predicted (Schaffer and Wegner, 1980; Abrahams, Anderson, Licciardello and Ramakrishnan, 1979) to be

$$\nu^{-1} = \varepsilon + O(\varepsilon^2). \tag{1.49}$$

When this result is extrapolated to three dimensions ($\varepsilon = 1$), the conductivity exponent

$$\mu = 1 + O(\varepsilon). \tag{1.50}$$

This early prediction was initially widely acclaimed as a plausible theoretical explanation for the critical behavoior commonly observed in several systems (e.g. compensated doped semiconductors, see below), where $\mu \approx 1$. From the theoretical side, this result was believed to be exact from more then ten years, since very tedious subsequent work established that higher order corrections to ν , when evaluated to second and even third order in ε all vanished! Very surprisingly, a tour-de-force calculation by Hikami (to $O(\varepsilon^5)$) succeeded in finding nonvanishing correction to this exponent, giving an estimate

$$\mu \approx 0.67\tag{1.51}$$

in three dimensions. This calculation demonstrated that the exact value ($\mu \approx 1.58$ for the "orthogonal ensemble" describing ordinary potential scattering) lies hopelessly far from the estimates based on the ε -expansion.

Current theories of the metal-insulator transition 39

This imporant result indicates a potentially serious shortcoming of any such perturbative RG schemes. As in ordinary (magnetic) critical phenomena, the ε -expansion around d = 2 seems to have extremely bad convergence properties (Castilla and Chakravarty, 1993) when extrapolated to d = 3, making it a virtually useless theoretical tool for making quantitative estimates for the critical exponents. The reason for generally poor convergence of $d = 2 + \varepsilon$ expansions is at present not fully understood, although related work (Kamal and Murthy, 1993) suggested that it reflects an inability to incorporate topological excitations ("hedgehog" configurations for the d = 3 Heisenberg model they studied) within any perturbative scheme. This behavior should be contrasted to that familar from the $4 - \varepsilon$ RG approaches in standard critical phenomena. Here, a consistent description of the critical point is found even at the mean-field level, which becomes exact for $d > d_{uc} = 4$. The fluctuation effects described by RG flows only provide the corrections to mean-field values for the critical exponents, and in general proved to have much superior convengence properties, often allowing surprisingly accurate estimates when extrapolated to d = 3. Interestingly, the complete lack of spatial correlations within mean-field theory allows for all spin configurations to be considered on equal footing within the $4-\varepsilon$ scheme, including both the smooth configurations described by perturbative methods, and the topological defects which they ignore.

Returning to metal-insulator transitions, one may speculate that the difficulties with peturbative approaches reflect that here the very existence of the phase transition emerges only due to fluctuation corrections; no transition is found at the mean-field (saddle-point) level describing Drude-Boltzmann theory. Developing a more appriate mean-field description of the MIT thus appears as one of the most promising directions for future work.

Generalized Fermi liquid theory for disordered electrons. In the mid 1980s, these ideas were extended with a great deal of effort in the formulation of a scaling theory of interacting disordered electrons by Finkelshtein (Finkel'stein, 1983) and many followers (Castellani, Castro, Lee and Ma, 1984; Belitz and Kirkpatrick, 1994). While initially shrouded by a veil of quantum field theory, these theories were later given a simple physical interpretation in terms of Fermi liquid ideas (Castellani, Kotliar and Lee, 1987; Kotliar, 1987) for disordered electrons. Technical details of these theories are of considerable complexity, and the interested reader is referred to the original literature (Finkel'stein, 1983; Finkel'stein, 1984; Castellani, Castro, Lee and Ma, 1984; Belitz and Kirkpatrick, 1994). Here we just summarize the principal results, in order to clarify the constraints imposed by these Fermi liquid approaches.

Within the Fermi liquid theory for disordered systems (Castellani, Kotliar and Lee, 1987; Kotliar, 1987), the low energy (low temperature) behavior of the system is characterized by a small number of effective parameters, which include the diffusion constant D, the frequency renormalization factor Z, and the singlet and triplet interaction amplitudes γ_s and γ_t . These quantities can also be related to the corresponding quasi-particle parameters which include the quasi-particle density of states

$$\rho_Q = Z\rho_o, \tag{1.52}$$

and the quasi-particle diffusion constant

$$D_Q = D/Z \sim D/\rho_Q. \tag{1.53}$$

Here, ρ_o is the "bare" density of states which describes the noninteracting electrons. In the absence of interactions, the single-particle density of states is only weakly modified by disorder and remains noncritical (finite) at the transition (Schaffer and Wegner, 1980).

Using these parameters, we can now express the thermodynamic response functions as follows. We can write the compressibility

$$\chi_c = \frac{dn}{d\mu} = \rho_Q [1 - 2\gamma_s], \qquad (1.54)$$

the spin susceptibility

$$\chi_s = \mu_B^2 \rho_Q [1 - 2\gamma_t], \tag{1.55}$$

and the specific heat

$$C_V = 2\pi^2 \rho_Q T/3. \tag{1.56}$$

In addition, we can use the same parameters to express transport properties such as the conductivity

$$\sigma = \frac{dn}{d\mu} D_c = \rho_Q D_Q, \tag{1.57}$$

as well as the density-density and spin-spin correlation functions

$$\pi(q,\omega) = \frac{dn}{d\mu} \frac{D_c q^2}{D_c q^2 - i\omega} \qquad \chi_s(q,\omega) = \chi_s \frac{D_s q^2}{D_s q^2 - i\omega}.$$
(1.58)

Here, we have expressed these properties in terms of the spin and charge diffusion constants, which are defined as

$$D_c = \frac{D}{Z(1-2\gamma_s)}; \quad D_s = \frac{D}{Z(1-2\gamma_t)}.$$
 (1.59)

Note that the quantity D is not the charge diffusion constant D_c that enters the Einstein relation, Eq. (1.57). As we can write $\sigma = \rho_o D$, and since ρ_o is not critical at any type of transition, the quantity D (also called the "renormalized diffusion constant") has a critical behavior identical to that of the conductivity σ . We also mention that the quasi-particle diffusion constant $D_Q = D/Z$ has been physically interpreted as the heat diffusion constant.

Weak-coupling renormalization group approach of Finkelshtein. The Fermi liquid relations provide constraints relating different Landau parameters, but they cannot predict their specific values, or how should they behave in the vicinity of the metalinsulator transition. In practical applications of these approaches to disordered system one assumes that the Landau parameters characterizing the clean Fermi liquid are known (say from experiments), and then examines how they are modified when impurities are added to the system. Precise prediction for these impurity effects have been obtained in the limit of weak disorder, where perturbative "quantum corrections" to all quantities have been calculated (Lee and Ramakrishnan, 1985). As in the noninteracting limit, singular logarithmic corrections are found in d = 2, signaling that a systematic weak coupling approach can be developed, to investigate the instability of the metallic phase under impurity scattering. While conceptually simple and transparent, practical calculations to implement this program proved of considerable complexity, even within the framework of weak coupling approaches.

Perturbative renormalization group calculations (Finkel'stein, 1983; Finkel'stein, 1984; Castellani, Castro, Lee and Ma, 1984; Belitz and Kirkpatrick, 1994) based on the $2+\varepsilon$ expansion have been used to make explicit predictions for the values of the critical exponents for different universality classes. The details of this formulation will not be elaborated here, as it has been discussed in great detail in several excellent reviews. Instead, we comment on the physical content of these theories, and indicate what aspects of the problem may and which may not be addressed using this framework.

Physical content of Finkelstein's theory. The mathematical complexity of of Finkelstein's formalism seems, at first sight, to shroud its physical content with a veil of mystery, and render it difficult to comprehent to all byt the very few specialists working in the field. Later works (Castellani, Castro, Lee and Ma, 1984; Castellani, Kotliar and Lee, 1987; Zala, Narozhny and Aleiner, 2001), however, succeeded to rederive most of Finkelstein's results using standard (albeit much less elegant) diagrammatic approaches, allowing for a simple and transparent physical interpretation. The main points include:

- This theory describes a disordered Fermi liquid, specifically the scale-dependence of the disorder-modified Landau parameters.
- As in the original scaling theory of localization, the transition is identified as an unstable fixed point for the conductance. If the interaction amplitudes retain finite (albeit renormalized) values at the fixed point, then the scaling scenario is essentially of the same form as for noninteracting localization. This behavior is found (Belitz and Kirkpatrick, 1994) for all the universality classes with broken time reversal invariance (BTRI): in presence of an external magnetic field, magnetic impurity scattering, or presence of spin-orbit scattering.
- The metallic phase, as in any Fermi liquid, retains the same qualitative behavior at T = 0 as a weakly disordered systems of noninetracting electrons. In particular, this means that all thermodynamic quantities (e.g. the spin susceptibility χ or the Sommerfeld coefficient $\gamma = C/T$) remain finite away from the transition.
- The scaling hypothesis, which is built into this formalism, guarantees that the low temperature behavior of all quantities is qualitatively identical within the each (metallic or insulating) phase. For example, the leading low temperature \sqrt{T} corrections, first predicted by using weak-disorder perturbation theory of Altshuler and Aronov, should retain their form throughout the metallic phase.

These results may lead - at least in principle - to a mathematically consistent description of the metal-insulator transition in d = 3. In two dimensions, where Anderson localization is expected to distroy any metallic phase, the situation is more subtle. An especially interesting and still controversial situation is found for the "generic" model, i.e. in absence of BTRI perturbations. Here, a qualitatively new behavior is found due

to interaction effects. The triplet interaction amplitude γ_t is found to grow withour limits under rescaling (i.e. as *T* is lowered), even at infinitesimal disorder (Finkel'stein, 1983; Finkel'stein, 1984). As a result, the "interaction" correction for the conductivity (which to leading order is proportional to γ_t) is found to produce an "antilocalization" effect, which is predicted to "overcome" the usual weak localization term responsible for the instability of the d = 2 metallic phase. From this point of view, Finkelstein's results open a consistent possibility that interactions may be able to stabilize the metallic phase at sufficiently weak disorder, thus leading to a sharp metal-insulator transition in d = 2.

Most remarkaby, recent experiments on a variety two-dimensional electron gases have revealed behavior quite suggestive of the existence of precisely such a transition (Abrahams, Kravchenko and Sarachik, 2001). This contriversial phenomenon, first discovered by Kravchenko in 1995 (Kravchenko, Mason, Bowker, Furneaux, Pudalov and D'Iorio, 1995), has reignited interest in this fundamental question. Its many facets continue to facinate theorists and experimentalists alike, but much still remains to be understood, as described in more detail in Chap. 2. Even from the purel theoretical point of view, however, the physical interpretation of what precisely happens in a 2Dweakly disordered Fermi liquid still remains puzzling and controversial. The main issue is how to understand the singular behavior of the triplet interaction amplitude γ_t , which is found to grow without bounds at low temperature. In disordered Fermi liquid it indicates enhanced spin susceptibility $\chi \sim \gamma_t$, so this result may signal some kind of magnetic instability of the metallic phase. Various interpretations have been proposed ranging from disorder-induced ferromagnetism (Belitz and Kirkpatrick, 1994), to disorder-induced local moment formation (Castellani, Castro, Lee and Ma, 1984; Milovanović, Sachdev and Bhatt, 1989; Paalanen, Graebner, Bhatt and Sachdev, 1998; Dobrosavljević, Kirkpatrick and Kotliar, 1992). While some progress has recently been made by Punnoose and Finkelstein (see Chapter 4), using large-N methods, the issue remains controversial and chalenging.

Difficulties with runaway RG flows. From a more general perspective, any solution with such "runaway" coupling constant immediately brings into question the credibility of perturbative RG approaches. It is interesting to recall that similar situations have been encoutered in several other problems, where disorder effects in interacting systems have proved difficult to describe using perturbative RG methods. The first such example is the famed random-field Ising model (RFIM) (Imry and Ma, 1975), where a perturbative arguments of Parisi and Sourlas (PS) suggested (Parisi and Sourlas, 1979) that the critical behavior of the disorder problem can be mapped to the corresponding clean problem in d-2 dimensions. Because the clean Ising model does not order in d = 1, this work suggested that infinitesimal disorder would be sufficient to destroy the ordered phase even in d = 3. Later rigorous by Imbrie (Imbrie, 1984) rigorously showed that a finite temperature ordering survives sufficiently weak disorder in d = 3, thus invalidating the PS mechanism, which was claimed to hold to all orders in perturbation theory. The puzzle was soon resolved by more careful functional renormalization group methods of Daniel Fisher (Fisher, 1985), who showed that the perturbative fixed point of PS proves unstable, leading to "runaway" RG flows. Subsequent work (Mezard and Young, 1992; Mezard and Monasson, 1994; C.De Dominicis, 1995) found evidence that this reflects the proliferation of metastable states in the RFIM, a phenomenon which proves "invisible" in perturbative treatment. Interestingly, much later work (Kirkpatrick and Belitz, 1994; Belitz and Kirkpatrick, 1995) attempted a non-pertubative (albeit uncontrolled) solution of Finkelstein's nonlinear σ -model by proposing a new saddle point solution, and a RG approach based on a $6 - \varepsilon$ expansion, finding RG flows of a structure very similar to the PS theory for the RFIM. The authors interpreted these results as evidence that metastability and non-perturbative glassy effects cannot be ignored within the Finkelstein's model any more then they could for the RFIM.



Fig. 1.15 In the disordered Fermi liquid picture, the leading low-temperature dependence of transport reflect elastic scattering off a renormalized, but temperature-dependent random potential (dashed line). At low temperatures (bottom), the potential wells "fill-up" with electrons; in presence of repulsive (Coulomb) interactions, the screened (renormalized) potential has reduced amplitude (dashed line), leading to effectively weaker disorder. As the temperature increases (top), electrons thermally activate (shown by arrows) out of the potential wells, reducing the screening effect. This physical mechanism, which operates both in the ballistic and in the diffusive regime (Zala, Narozhny and Aleiner, 2001), is at the origin of all "quantum corrections" found within the Fermi liquid picture. It is dominant, provided that inelastic electron-electron scattering can be ignored. While this approximation is well justified in good metals, inelastic scattering (star symbol) is considerably enhanced in presence of strong correlation effects, often leading to disorder-driven non-Fermi liquid behavior (Miranda and Dobrosavljevic, 2005) and electronic Griffithe phases (Miranda and Dobrosavljevic, 2005) and electronic Griffithe phases (Miranda and Dobrosavljevic, 2009).

Yet another example of non-perturbative disorder effects has been found in recent studies of Hertz-Millis (Hertz, 1976; Millis, 1993; Sachdev, 2011) models for itinerant quantum criticality. Here early works attempted a perturbative RG treatment for weak disorder, only to find runaway flows. Later works by Vojta and Hoyos (Hoyos, Kotabage and Vojta, 2007), which used a complementary "strong-disorder renormalization group" (SDRG) methods, deminstrated that for these models disorder completely changes the physical nature of quantum criticality, leading to so-called "infiniterandomness fixed point" (IRFP) behavior (Miranda and Dobrosavljevic, 2005; Vojta, 2006). In such cases, the behavior of the system is deminated by rare disorder realizations, leading to the formation of Quantum Griffiths Phases (QGP) surrounding the critical point.

We should stress that these non-perturbative disorder effects do not bring into question the general ideas of scaling, but they they do seem to raise concerns about the application of weak coupling RG methods. In cases such as the IRFP universality class, the renormalized effective action describing the critical point assumes a qualitatively different form then that of the clean system. This simple fact makes it abundantly clear how and why perturbative methods may fail in a most dramatic fashion. Whether a similar mechanism resolves the puzzles identified by the perturbative solution of the Finkelstein model remains to be established by future work.

Conceptual limitations of the disordered Fermi liquid picture. Despite a great deal of effort invested in such calculations, the predictions of the perturbative RG approaches have met only limited success in explaining the experimental data in the critical region of the metal-insulator transition. We should emphasize, though, that limitations associated with these weak-coupling theories do not invalidate the potential applicability of Fermi liquid ideas *per se.* Having this in mind, it is also important to understand the fundamental limitations of the disordered Fermi liquid picture which is the underpinning of Finkelstein's field theory. The important physical conditions implicitly assumed by this formulation include:

- It describes leading low-temperature excitations, which are adiabatically connected to a noninteracting (but disordered) electronic system. These excitations, therefore, assume fermionic character. Other (spin or charge) collective excitations are assumed to play a subleading role in this temperature regime.
- The fermionic excitations are assumed to be sufficiently dilute, such that inelastic electron-electron scattering processes can be neglected. The leading low temperature corrections then reflect elastic scattering processes on a static, but temperature-dependent renormalized disorder potential. Upon disorder averaging, this can produce different temperature dependence in the diffusive and ballistic regimes. As explained in recently by Zala *et al.* (Zala, Narozhny and Aleiner, 2001), however, the basic physical mechanism for all these corrections is one and the same (Fig. 1.15). We should emphasize that these processes dominate only at sufficiently low temperatures, typically being only a small fraction of the Fermi temperature. In presence of strong electronic correlations, the "coherence temperature" T^* below which the Fermi liquid picture applies, may be very low; a good example are so-called "heavy-fermion" systems, where $T^* \ll T_F$ and a

very broad incoherent metallic regime is found. In presence of disorder, the coherence temperature T^* may be further reduced or even driven to zero (Miranda and Dobrosavljevic, 2005). This situation is illustrated in certain exactly solvable models with strong correlation and disorder (Dobrosavljević and Kotliar, 1993; Tanasković, Miranda and Dobrosavljević, 2004), which can be exatly solved in the limit of large coordination. Here strong disorder fluctuations stabilize "non-Fermi liquid" metallic behavior at any non-zero temperature.

- The standard formulation of the disordered Fermi liquid theory is derived at weak disorder, where the electrons are well delocalized accross the entire sample. The Landau interaction amplitude are assumed to be self-averaging, and are thus replaced by their averaged values. At stronger disorder, very strong spatial flauctuations may emerge (Miranda and Dobrosavljevic, 2005), so that the interaction effects may be much more pronounced in certain regions of the sample. These effects are most important in strongly correlated electronic systems, where the assumption of self-averaging may completely break down, leading to the formation of "electronic Griffiths phases" (Miranda and Dobrosavljević, 2001; Tanaskovic, Dobrosavljevic and Miranda, 2005; Andrade, Miranda and Dobrosavljevic, 2009). If this happens, then the simiplified version of disordered Fermi liquid theory may prove insufficient or even misleading.
- The Fermi liquid pictures assumes a unique ground state, implicitly ignoring the possibility metastable states resulting from the competition of disorder and the long-range Coulomb interactions. The associated electron glass behavior (Pastor and Dobrosavljević, 1999) is a well establised feature of disordered insulators, but its possible role in the critical regime (Dobrosavljević, Tanasković and Pastor, 2003b) has been ignored by the weak coupling approaches.

1.3.3 Order-parameter approaches to interaction-localization

Need for an order-parameter theory: experimental clues. In conventional critical phenomena, simple mean-field approaches such as the Bragg-Williams theory of magnetism, or the Van der Waals theory for liquids and gases work remarkably well everywhere except in a very narrow critical region (Goldenfeld, 1992). Here, effects of long wavelength fluctuations emerge that modify the critical behavior, and its description requires more sophisticated theoretical tools, based on renormalization group (RG) methods. A basic question then emerges when looking at experiments: is a given phenomenon a manifestation of some underlying mean-field (local) physics, or is it dominated by long-distance correlations, thus requiring an RG description? The answer for conventional criticality is well know, but how about metal-insulator transitions? Here the experimental evidence is much more limited, but we would like to emphasize a few well-documented examples which stand out.

• Doped semiconductors such as Si:P. These are the most carefully studied (Shklovskii and Efros, 1984) examples of the MIT critical behavior. Here the density-dependent conductivity extrapolated to T = 0 shows sharp critical behavior (M. A. Paalanen, 1991) of the form $\sigma \sim (n - n_c)^{\mu}$, where the critical exponent $\mu \approx 1/2$ for uncompensated samples (half-filled impurity band), while dramatically different $\mu \approx 1$ is found for heavily compensated samples of Si:P,B, or in presence of strong

magnetic fields. Most remarkably, the dramatically differences between these cases is seen over an extremely broad concentration range, roughly up to several times the critical density. Such robust behavior, together with simple (apparent values for the critical exponents, seems reminiscent of standard mean-field behavior in ordinary criticality.



Fig. 1.16 Critical behavior of the conductivity for uncompensated Si : P and compensated Si:P,B (M. A. Paalanen, 1991). The conductivity exponent $\mu \approx 1/2$ in absence of compensation, while $\mu \approx 1$ in its presence. Clearly distinct behavior is observed in a surprisingly broad range of densities, suggesting mean-field scaling. Since compensation essentially corresponds to carrier doping away from a half-filled impurity band (Shklovskii and Efros, 1984), it has been suggested (Lee and Ramakrishnan, 1985) that the difference between the two cases may reflect the role of strong correlations.

• Two-dimensional metal-insulator transitions. Signatures of a remarkably sharp metal-insulator transition has also been observed (Kravchenko, Mason, Bowker, Furneaux, Pudalov and D'Iorio, 1995; Popović, Fowler and Washburn, 1997; Abrahams, Kravchenko and Sarachik, 2001) in several examples of two-dimensional electron gases (2DEG) such as silicon MOSFETs. While some controversy regarding the nature or even the driving force for this transition remains a subject of intense debate, several experimental features seem robust properties common to most studied samples and materials. In particular, various experimental groups have demonstrated (Kravchenko, Mason, Bowker, Furneaux, Pudalov and D'Iorio, 1995; Popović, Fowler and Washburn, 1997) striking scaling of the resistivity curves in the critical region, which seems to display (Simonian, Kravchenko

and Sarachik, 1997) remarkable mirror symmetry ("duality") (Dobrosavljević, Abrahams, Miranda and Chakravarty, 1997) over a surprisingly broad interval of parameters. In addition, the characteristic behavior extends to remarkably high temperatures, which are typically *comparable the Fermi temperature* (Abrahams, Kravchenko and Sarachik, 2001). One generally does not expect a Fermi liquid picture of diluted quasiparticles to apply at such "high energies", or any correlation length associated with quantum criticality to remain long.



Fig. 1.17 The resistivity curves (left panel) for a two-dimensional electron system in silicon (Kravchenko, Mason, Bowker, Furneaux, Pudalov and D'Iorio, 1995) show a dramatic metal-insulator crossover as the density is reduced below $n_c \sim 10^{11} cm^{-2}$. Note that the system has "made up its mind" whether to be a metal or an insulator even at surprisingly high temperatures $T \sim T_F \approx 10K$. The right panel displays the scaling behavior which seems to hold over a comparable temperature range. The remarkable "mirror symmetry" (Simonian, Kravchenko and Sarachik, 1997) of the scaling curves seems to hold over more then an order of magnitude for the resistivity ratio. This surprising behavior has been interpreted (Dobrosavljević, Abrahams, Miranda and Chakravarty, 1997) as evidence that the transition region is dominated by strong coupling effects characterizing the insulating phase.

• *High temperature anomalies - Mooij correlation.* Surprisingly similar metal-insulator crossover is seen at elevated temperatures in several systems (Lee and Ramakrishnan, 1985) where the MIT is driven by increasing the level of disorder at fixed electron density. This behavior, first identified by Mooij (Mooij, 1973), has recently been studied in phase change materials (Siegrist, Jost, Volker, Woda, Merkelbach, Schlockermann and Wuttig, 2011). Here, the temperature coefficient of the resistivity (TCR) is found to change sign, indicating the crossover from metallic $(d\rho/dT > 0)$ to insulating(-like) $(d\rho/dT > 0)$ transport, around the values of the resistivity close to the "Mott limit" $\rho_c = 1/\sigma_{min}$ (Fig. 1.18). This behavior is consistent with the early ideas of Mott, suggesting that in metals, as temperature is increased, the resistivity should display resistivity saturation (Fisk and Webb, 1976) as soon as the mean-free path approaches the atomic scale. Although reminiscent to what is seen in (d = 2) silicon MOSFETs, this behavior observed in bulk (d = 3) systems is inconsistent with the expectations based on the scaling theories of localization (e.g. a divergent critical resistivity $\rho_c(T) \sim 1/T^{\frac{d-2}{z}}$). Since any concievable coherence length must be short at such elevated temperatures, local (incoherent) scattering processes are likely to be at the origin of this puzzling behavior. This should be contrasted to the well-known weak localization and interaction corrections in a disordered Fermi liquid, processes dominated by coherent miltiple-scattering processes at long lengthscales.



Fig. 1.18 High-temperature transport in the disorder-driven metal-insulator transition region in recently discovered "phase change" materials (Siegrist, Jost, Volker, Woda, Merkelbach, Schlockermann and Wuttig, 2011). Here disorder is tuned at fixed carrier concentration by partially annealing the parent amorphous insulator. The change of the sign of TCR is found to occur precisely at the "separatrix" coinciding with the estimated Mott limit.

• High temperature violations of the Mott limit. Examples of metallic $(d\rho/dT > 0)$ transport with resistivities dramatically exceeding the Mott limit have been reported only in sufficiently clean systems not too far from the Mott insulating state. This curious behavior was first noted shortly after the discovery of cuprate superconductors (Hussey, Takenaka and Takagi, 2004), and was quickly interpreted as a "smoking gun" of non-Fermi liquid physics and strong correlation effects. Further examples have been documented in organic Mott systems (Limelette, Wzietek, Florens, Georges, Costi, Pasquier, Jerome, Meziere and Batail, 2003b) and transition metal oxides such as V_2O_3 (Limelette, Georges, Jerome, Wzietek, Metcalf and Honig, 2003a). In all these cases, the violation of the Mott limit was found at temperatures exceeding the Fermi-liquid coherence scale T^* , and was observed to coincide with the suppression of the corresponding Drude peak in the optical conductivity. Similarly to resistivity saturation, such "bad metal" behavior seems to emerge only in the high temperature incoherent regime, where local scattering processes dominate.



Fig. 1.19 Transport behavior in a strongly correlated metal $\kappa - (BETD - TTF)_2Cu[N(CN)_2]Cl$, approching the pressure-driven Mott transition. Here, Fermi liquid transport ($\rho \sim T^2$) is found only at the lowest temperatures. The unusual transport behavior at higher temperatures, displaying pronounced resistivity maxima well exceeding the Mott limit, has been quantitatively explained using DMFT theories (Limelette, Wzietek, Florens, Georges, Costi, Pasquier, Jerome, Meziere and Batail, 2003*b*; Radonjic, Tanaskovic, Dobrosavljevic and Haule, 2010).

All these experiments taken together provide strong hints that in many systems of current interest an appropriate mean-field description is what is needed. It should provide the equivalent of a Van der Waals equation of state, for the metal-insulator transition problem of disordered interacting electrons. Unfortunately, such a theory has long been elusive, primarily due to a lack of a simple order-parameter formulation for this problem. Very recently, an alternative approach to the problem of disordered interacting electrons has been developed, based on dynamical mean-field (DMFT) methods (Georges, Kotliar, Krauth and Rozenberg, 1996). This formulation is largely complementary to the scaling approach, and has already resulting in several striking predictions. In the following, we briefly describe this method, and summarize the main results that have been obtained so far.

The DMFT physical picture. The main idea of the DMFT approach is in principle very close to the original Bragg-Williams (BW) mean-field theories of magnetism (Goldenfeld, 1992). It focuses on a single lattice site, but replaces (Georges, Kotliar, Krauth and Rozenberg, 1996) its environment by a self-consistently determined "effective medium", as shown in Fig. 1.3.



Fig. 1.20 In dynamical mean-field theory, the environment of a given site is represented by an effective medium, represented by its "cavity spectral function" $\Delta_i(\omega)$. In a *disordered* system, $\Delta_i(\omega)$ for different sites can be very different, reflecting Anderson localization effects (Anderson, 1958).

In contrast to the BW theory, the environment cannot be represented by a static external field, but instead must contain the information about the dynamics of an electron moving in or out of the given site. Such a description can be made precise by formally integrating out all the degrees of freedom on other lattice sites. In presence of electron-electron interactions, the resulting local effective action has an arbitrarily complicated form. Within DMFT (Georges, Kotliar, Krauth and Rozenberg, 1996), the situation simplifies, and all the information about the environment is contained in the local single particle spectral function $\Delta_i(\omega)$. The calculation then reduces to solving an appropriate quantum impurity problem supplemented by an additional selfconsistency condition that determines this "cavity function" $\Delta_i(\omega)$.

The precise form of the DMFT equations depends on the particular model of interacting electrons and/or the form of disorder, but most applications (Georges, Kotliar, Krauth and Rozenberg, 1996) to this date have focused on Hubbard and Anderson lattice models. The approach has been very successful in examining the vicinity of the Mott transition in clean systems, and it has met spectacular successes in elucidating various properties of several transition metal oxides, heavy fermion systems and Kondo insulators (Kotliar, Savrasov, Haule, Oudovenko, Parcollet and Marianetti, 2006).

The central quantity in the DMFT approach is the local "cavity" spectral function $\Delta_i(\omega)$. From the physical point of view, this object essentially represents the *available electronic states* to which an electron can "jump" on its way out of a given lattice site. As such, it provides a natural order parameter description for the MIT (Dobrosavljević and Kotliar, 1998). Of course, its form can be substantially modified by either the electron-electron interactions or disorder, reflecting the corresponding modifications of the electron dynamics. According to Fermi's Golden Rule, the transition rate to a neighboring site is proportional to the density of final states - leading to insulating behavior whenever $\Delta_i(\omega)$ has a gap at the Fermi energy. In the case of a Mott transition in the absence of disorder, such a gap is a direct consequence of the strong on-site Coulomb repulsion, and is the same for every lattice site.



Fig. 1.21 The local cavity spectral function $\Delta_i(\omega)$ as the order parameter for the MIT. In a metal (a) there are available electronic states near the Fermi level (dashed line) to which an electron from a given site can delocalize. Both for a Mott insulator (b) and the Anderson insulator (c) the Fermi level is in the gap, and the electron cannot leave the site. Note that the *averaged* spectral function (dotted line in (c)) has no gap for the Anderson insulator, and thus cannot serve as an order parameter.

The situation is more subtle in the case of disorder-induced localization, as first noted in the pioneering work of Anderson (Anderson, 1958). Here, the *average* value of $\Delta_i(\omega)$ has no gap and thus cannot serve as an order parameter. However, as Anderson noted a long time ago, "...no real atom is an average atom..." (Anderson, 1978). Indeed, in an Anderson insulator, the environment "seen" by an electron on a given site can be very different from its average value. In this case, the *typical* "cavity" spectral function $\Delta_i(\omega)$ consists of several delta-function (sharp) peaks, reflecting the existence of localized (bound) electronic states (Fig. 1.21). Thus a *typical* site is embedded in an environment that has a *gap* at the Fermi energy - resulting in insulating behavior. We

emphasize that the location and width of these gaps strongly vary from site to site. These strong fluctuations of the local spectral functions persist on the metallic side of the transition, where the typical spectral density $\Delta_{typ} = \exp \langle \ln(\Delta_i) \rangle$ can be much smaller than its average value (Dobrosavljević, Pastor and Nikolić, 2003*a*). Clearly, a full distribution function is needed (Anderson, 1978) to characterize the system. The situation is similar as in other disordered systems, such as spin glasses (Mézard, Parisi and Virasoro, 1986). Instead of simple averages, here the entire distribution function plays a role of an order parameter, and undergoes a qualitative change at the phase transition. The DMFT formulation thus naturally introduces self-consistently defined dynamical order parameters, that can be utilized to characterize the qualitative differences between various phases. In presence of disorder, these order parameters have a character of distribution functions, which change their qualitative form as we go from the normal metal to the non-Fermi liquid metal, to the insulator.

Physical content of DMFT approaches. DMFT and its various generalizations are designed to capture strong but local correlation effects. Here we will not discuss the technical details on these approaches, which will be discussed, in some detail, in Chapter 6. Instead, we give a brief summary of the advantages and the limitations of the existing DMFT theories, in applications to strongly correlated electronic systems with and without disorder.

- The approach is formally exact in the limit of large coordination, but in general it represents a "conserving" approximation scheme (Kotliar, Savrasov, Haule, Oudovenko, Parcollet and Marianetti, 2006), in the sense of Baym and Kadanoff. Although it reproduces Fermi liquid behavior of metals at low temperature, it is not restricted to this regime. In fact, DMFT is more accurate (essentially exact) in the high-temperature incoherent regime, because this is where any spatiall correlations ignored by DMFT become negligible.
- The local correlation effects are described through the local self-energies $\Sigma_i(\omega)$, defining both the quasiparticle effective mass $m^* = 1 (\partial \Sigma_i / \partial \omega)_{\omega=0}$, and the inelastic scattering rate $\hbar \tau_{in}^{-1}(\omega) = -Im\Sigma_i(\omega=0)$. In presence of disorder these quantities display spatial fluctuations (Miranda and Dobrosavljevic, 2005), and need to be characterized with appropriate distribution functions.
- DMFT cannot be used to properly describe those phenomena which are dominated by long-wavelength spatial fluctuations. It cannot provide a description of anomalous dimensions of various physical quantities within the narrow "Ginzburg" region around a critical point, at least for ordinary criticality. On the other hand, recent work suggested the possibility of "Local Quantum Criticality" (Si, Rabello, Ingersent and Smith, 2001), where DMFT-like approaches may represent the proper description.

We should stress that the local dynamical description description provided by DMFT is a feature built-in from the start. How broad is the range of parameters and models where this approximation is valid is not obvious or precisely known at this time, despite the many successful applications (Kotliar, Savrasov, Haule, Oudovenko, Parcollet and Marianetti, 2006) of DMFT to various experimental systems. Interestingly, very recent work based on "holographic duality" (Sachdev, 2010; McGreevy, 2010) ideas borrowed from super-string theory suggests that certain strong-coupling classes of QCPs may exist, having strictly local character. If these arguments prove correct, they may provide fundamental insight into why and where the local ideas implied by DMFT can be expected to apply, and in which cases the do not.

Applications and extensions of DMFT. The first practical application of DMFT has focused at the Mott transition in a single-band Hubbard model, sheding new light on systems ranging from transition metal oxides to organic Mott systems. Even in this simplest "single-site" version, the method can be easily extended to multi-band and multy-orbital models, features that are of key importance in materials such as rareearth intermetallics (heavy fermion systems) and the recently discovered familty of iron-pnictides (Basov and Chubukov, 2011). The technical difficulties in applying the methods to multi-orbital systems has pleagued early studies, essentially due to lack of a reliable "quantum impurity solver" for DMFT equations. Very recent progress based on "continuos time quantum Monte Carlo" methods (Gull, Millis, Lichtenstein, Rubtsov, Troyer and Werner, 2011) has provided a practical and efficient solution in many cases, opening avenues for applications to many materials.

Other complications surrounded efforts to dovetail the DMFT methods for strong correlation with first-principle density-functional electronic structure methods (DFT). Considerable efforts have been invested in this program (Kotliar, Savrasov, Haule, Oudovenko, Parcollet and Marianetti, 2006), and over the last few years combined density functional and DMFT studies have bloomed into a veritable industry, resulting in impressive and accurate first-principles descriptions of many strongly correlated materials. Finally, effects of inter-site correlations have also been a focus of much recent work, leading to various "cluster" generalizations of DMFT, resulting in substantial new insight (Haule and Kotliar, 2007; Yang, Fotso, Su, Galanakis, Khatami, She, Moreno, Zaanen and Jarrell, 2011) in systems such as high Tc cuprate superconductors. Different aspect of these theories have been discussed in detail in several recent reviews (Maier, Jarrell, Pruschke and Hettler, 2005; Kotliar, Savrasov, Haule, Oudovenko, Parcollet and Marianetti, 2006). In the following we focus on applications of DMFT to disordered systems (Miranda and Dobrosavljevic, 2005), and list the physical phenomena and regimes that have found a natural description within the DMFT framework, but which remain difficult to address using complementary approaches.

The simplest DMFT theories focus on the dynamical effects of local interactions, while describing the environment of a given site as an average "effective medium". In presence of disorder, this formulation reduces (Dobrosavljević and Kotliar, 1994) to the well-known "coherent potential approximation" (CPA) in the noninteracting limit. This theory, while providing a reasonable description of disorder-averaged singleparticle spectral functions, is unable to capture Anderson localization, as well as other imporant disorder effects such as the formation of inhomogeneous or glassy phases. Recent efforts, however, have resulted in various extensions of DMFT, which have succeeded to incorporate various aspects of such disorder-driven phenomena, as follows.

- "Statistical DMFT" theories (Dobrosavljević and Kotliar, 1997) which calculate the local site-dependent self-energies for a given fixed realization of disorder. Here the strong correlation effects are described in DMFT fashion, while disorder fluctuations are treated in an exact numerical approach. In a sense, this method can be regarded as a quantum generalization of the "TAP equation" (Thouless, Anderson and Palmer, 1977) approach to classical spin glasses. In applicatio to disordered Huibbard and Anderson lattrice models, it led to the discovery of disorder-driven non-Fermi liquid behavior (Miranda and Dobrosavljevic, 2005) and "Electronic Griffiths phases" (Miranda and Dobrosavljević, 2001; Tanasković, Miranda and Dobrosavljević, 2004; Andrade, Miranda and Dobrosavljevic, 2009).
- "*Typical-Medium Theories*" (Dobrosavljević, Pastor and Nikolić, 2003*a*) which treat both the interactions and disorder fluctuations in a DMFT-like self-consistent fashion. This method is the simplest order-parameter theory of Mott-Anderson localization.
- "Exteded DMFT (EDMFT)" theories discribing the effect of the bosonic collective modes due to inter-site interactions. The EDMFT approach, which was much utilized in recent work on quantum criticality in Kondo systems (Smith and Si, 2000), also proved very effective in describing quantum spin glass (Sachdev, Read and Oppermann, 1995), and especially in describing the effects of the long-ranged Coulomb interaction (Chitra and Kotliar, 2000). It provided a quantitatively accurate description of the pseudogap phenomena in Coulomb systems (Pankov and Dobrosavljević, 2005), and clarified the relation between glassy freezing and formation (Pastor and Dobrosavljević, 1999) of the universal Efros-Shklovskii Coulomb gap in presence of disorder.

1.3.4 Conclusions and outlook

Recent years have seen considerable progres in finding fascinating but often baffling examples of materials that seem to belong to the metal-insulator transition region. In many of these examples, the electron localization does not appear to have a conventional "band transition" character, where some kind of uniform ordering leads to a band gap opening at the Fermi surface. Instead, mechanisms such as Mott and Aderson localization are invoqued, processes that simply do not fit the cherished mold of spontaneous symmtry breaking. Neverthless, in systems ranging from heavy doped semiconductors, dilute two-dimensional electron gases, to organic charge-transfer salts, to cuprate-oxide and iron-pnictide materials, the metal-insulator transition region typically assumes the form expected of quantum criticality.

The traditional approaches to the metal-insulator transition, dating to the 1980s, tried to circumvent the absence of an obvious order-parameter description by systematically examining the stability of the metallic (Fermi liquid) phase to weak disorder. This approach, despite its formal elegance and conceptual simplicity, did not find many convincing applications, most likely because its inherent inability to describe genuine strong correlation effects - the role of "Mottness". Indeed, much of the efforts of the theoretical condensed matter community over the last twenty years has focused on the Mott physics in its many forms. One central physics question has emerged from all these studies. This issue, first emphasized in the pioneering works of Mott and Anderson, relates to what are the relevant length-scales that dominate the "bad-metal" regime. The perspective provided by the Fermi liquid picture of metals and the symmetry breaking paradigm of conventional criticality both suggest that long lengthscales should hold the key. However, as hinted by the title of Phillip Anderson's 1978 Nobel Prize lecture "Local Moments and Localized States" (Anderson, 1978), sufficiently close to insulating states, the real-space representation may offer a better starting point.

Over the last twenty years, these local ideas acquired a precise and systematic language with the rise of DMFT approaches. This provided a natural *dynamical* orderparameter description for strongly correlated systems with and without disorder. The progress made by the various forms and extensions of DMFT is very encouraging, since they proved capable of incorporating all the basic mechanisms for electron localization. Many of these phenomena, such as the emergence of strongly inhomogneous phases, or the description of glassy dynamics, proved to be beyond the scope of conventional Fermi-liqui based theories of interaction-localization. In all fairness, though, the problem remains remains far from being resolved, and much more focused effort will be neccessary to combine all the facets of these fasciating theories in a comprehensive and well established picture of the metal-insulator transition region.

1.4 Acknowledgments

This work was supported by the NSF through grant DMR-1005751, and by the National High Magnetic Field Laboratory.

References

- Abrahams, E., Anderson, P. W., Licciardello, D. C., and Ramakrishnan, T. V. (1979). *Phys. Rev. Lett.*, **42**, 673–676.
- Abrahams, E., Kravchenko, S. V., and Sarachik, M. P. (2001). *Rev. Mod. Phys.*, **73**, 251–266.
- Abrikosov, A. A., Gor'kov, L. P., and Dzyaloshinskii, I. E. (1975). Methods of Quantum Field Theory in Statistical Physics. Dover, New York.
- Altshuler, B. L. and Aronov, A. B. (1979). JETP Lett., 50, 968.
- Amaricci, A., Camjayi, A., Haule, K., Kotliar, G., Tanasković, D., and Dobrosavljević, V. (2010, Oct). *Phys. Rev. B*, 82(15), 155102.
- Anderson, P. W. (1958). Phys. Rev., 109, 1492–1505.
- Anderson, P. W. (1959, Jul). *Phys. Rev.*, **115**(1), 2.
- Anderson, P. W. (1978). Rev. Mod. Phys., 50(2), 191.
- Ando, T., Fowler, A. B., and Stern, F. (1982). Rev. Mod. Phys., 54, 437-672.
- Andrade, E. C., Miranda, E., and Dobrosavljevic, V. (2009). Physical Review Letters, 102, 206403.
- Ashcroft, N. W. and Mermin, D. (1976). *Solid State Physics*. Thomson Learning, Toronto.
- Basov, D. and Chubukov, A. (2011). Nature Physics, 7, 272–276.
- Belitz, D. and Kirkpatrick, T. R. (1994). Rev. Mod. Phys., 66, 261.
- Belitz, D. and Kirkpatrick, T. R. (1995). Phys. Rev. B, 52, 13922.
- Beloborodov, I. S., Lopatin, A. V., Vinokur, V. M., and Efetov, K. B. (2007, Apr). *Rev. Mod. Phys.*, **79**(2), 469–518.
- Berker, A. N. (1991). J. Appl. Phys., 70, 5941.
- Bogdanovich, S. and Popović, D. (2002). Phys. Rev. Lett., 88, 236401.
- Brinkman, W. F. and Rice, T.M. (1970). *Phys. Rev. B*, 2, 4302.
- Camjayi, A., Haule, K., Dobrosavljevic, V., and Kotliar, G. (2008). *Nature Physics*, 4, 932.
- Cândido, Ladir, Bernu, B., and Ceperley, D. M. (2004). *Physical Review B*, **70**, 094413.
- Cândido, Ladir, Phillips, Philip, and Ceperley, D. M. (2001, Jan). *Phys. Rev. Lett.*, **86**(3), 492–495.
- Casey, A., Patel, H., Nyéki, J., Cowan, B. P., and Saunders, J. (2003). Phys. Rev. Lett, 90, 115301.
- Castellani, C., Castro, C. Di, Lee, P. A., and Ma, M. (1984). Phys. Rev. B, 30, 527.
- Castellani, C., Kotliar, B. G., and Lee, P. A. (1987). Phys. Rev. Lett., 56, 1179.
- Castilla, Guillermo E. and Chakravarty, Sudip (1993, Jul). *Phys. Rev. Lett.*, **71**(3), 384–387.
- C.De Dominicis, H.Orland, T.Temisvari (1995). J. de Physique I, 5, 987.

Chen, Yong P., Sambandamurthy, G., Wang, Z. H., Lewis, R. M., Engel, L. W., Tsui,
D. C., Ye, P. D., Pfeiffer, L. N., and West, K. W. (2006). *Nature Physics*, 2, 452.
Chitra, R. and Kotliar, G. (2000). *Phys. Rev. Lett.*, 84, 3678.

Clark, B. K., Casula, M., and Ceperley, D. M. (2009, Jul). *Phys. Rev. Lett.*, **103**(5),

- 055701. Departe F (2002) Nanagada Phaga Sar
- Dagotto, E. (2002). Nanoscale Phase Separation and Colossal Magnetoresistance. Springer-Verlag, Berlin.
- Dagotto, Elbio (2005). Science, **309**, 257.
- Dalidovich, D. and Dobrosavljević, V. (2002). Phys. Rev. B, 66, 081107.
- Dobrosavljević, V., Abrahams, E., Miranda, E., and Chakravarty, S. (1997). *Phys. Rev. Lett.*, **79**, 455.
- Dobrosavljević, V., Kirkpatrick, T. R., and Kotliar, B. G. (1992). *Phys. Rev. Lett.*, **69**, 1113.
- Dobrosavljević, V., Kirkpatrick, T. R., and Kotliar, G. (1992). *Phys. Rev. Lett.*, **69**, 1113.
- Dobrosavljević, V. and Kotliar, G. (1993). Phys. Rev. Lett., 71, 3218.
- Dobrosavljević, V. and Kotliar, G. (1994). Phys. Rev. B, 50, 1430.
- Dobrosavljević, V. and Kotliar, G. (1997). Phys. Rev. Lett., 78, 3943.
- Dobrosavljević, V. and Kotliar, G. (1998). Phil. Trans. R. Soc. Lond. A, 356, 1.
- Dobrosavljević, V., Pastor, A.A., and Nikolić, Branislav K. (2003*a*). *Europhys. Lett.*, **62**, 76–82.
- Dobrosavljević, V., Tanasković, D., and Pastor, A. A. (2003b). Phys. Rev. Lett., 90, 016402.
- Efros, A. L. and Shklovskii, B. I. (1975). J. Phys. C, 8, L49.
- Finkel'stein, A. M. (1983). Zh. Eksp. Teor. Fiz., 84, 168. [Sov. Phys. JETP 57, 97 (1983)].
- Finkel'stein, A. M. (1984). Zh. Eksp. Teor. Fiz., 86, 367. [Sov. Phys. JETP 59, 212 (1983)].
- Fisher, Daniel S. (1985, Jun). *Phys. Rev. B*, **31**(11), 7233–7251.
- Fisk, Z. and Webb, G. W. (1976, May). Phys. Rev. Lett., 36(18), 1084–1086.
- Flouquet, J. (2005). Volume 15, Progress in Low Temperature Physics, pp. 139 281. Elsevier.
- Georges, A., Kotliar, G., Krauth, W., and Rozenberg, M. J. (1996). *Rev. Mod. Phys.*, **68**, 13.
- Goldenfeld, N. (1992). Lectures on phase transitions and the renormalization group. Addison-Wesley, Reading.
- Gor'kov, L. P. and Sokol, A. V. (1987). JETP Lett., 46, 420.
- Grannan, Eric R. and Yu, Clare C. (1993, Nov). *Phys. Rev. Lett.*, **71**(20), 3335–3338. Grüner, G. (2000). *Density waves in solids*. Frontiers in Physics. Perseus Pub.
- Gull, Emanuel, Millis, Andrew J., Lichtenstein, Alexander I., Rubtsov, Alexey N., Troyer, Matthias, and Werner, Philipp (2011, May). *Rev. Mod. Phys.*, **83**(2), 349–404.
- Haule, Kristjan and Kotliar, Gabriel (2007, Sep). Phys. Rev. B, 76(9), 092503.

Hertz, J. A. (1976). Phys. Rev. B, 14, 1165.

Hewson, A. C. (1993). The Kondo Problem to Heavy Fermions. Cambridge University

58 References

Press, Cambridge.

- Hoyos, José A., Kotabage, Chetan, and Vojta, Thomas (2007, Dec). *Phys. Rev. Lett.*, **99**(23), 230601.
- Hubbard, J. (1963). Proc. R. Soc. (London) A, 276, 238.
- Hussey, N. E., Takenaka, K., and Takagi, H. (2004). *Philosophical Magazine*, **84**(27), 2847–2864.
- Imbrie, J. Z. (1984). Phys. Rev. Lett., 53, 1747.
- Imry, Y. and Ma, S. K. (1975). Phys. Rev. Lett., 35, 1399.
- Jamei, R., Kivelson, S., and Spivak, B. (2005). Phys. Rev. Lett., 94, 056805.
- Jaroszyński, J., Andrearczyk, T., Karczewski, G., Wróbel, J., Wojtowicz, T., Popović, Dragana, and Dietl, T. (2007). *Physical Review B*, **76**(4), 045322.
- Jaroszynski, J., Popovic, Dragana, and Klapwijk, T. M. (2004). *Phys. Rev. Lett.*, **92**, 226403.
- Kamal, Michael and Murthy, Ganpathy (1993, Sep). *Phys. Rev. Lett.*, **71**(12), 1911–1914.
- Kirkpatrick, S. (1973). Rev. Mod. Phys., 45, 574.
- Kirkpatrick, T. R. and Belitz, D. (1994). Phys. Rev. Lett., 74, 1178.
- Kohsaka, Y., Iwaya, K., Satow, S., Hanaguri, T., Azuma, M., Takano, M., and Takagi, H. (2004). Phys. Rev. Lett., 93, 097004.
- Kotliar, G. (1987). Anderson Localization. Springer, Berlin.
- Kotliar, G. and Ruckenstein, A. E. (1986). Phys. Rev. Lett., 57(11), 1362.
- Kotliar, G., Savrasov, S. Y., Haule, K., Oudovenko, V. S., Parcollet, O., and Marianetti, C. A. (2006, Aug). Rev. Mod. Phys., 78(3), 865–951.
- Kravchenko, S. V., Mason, W. E., Bowker, G. E., Furneaux, J. E., Pudalov, V. M., and D'Iorio, M. (1995). *Phys. Rev. B*, **51**, 7038.
- Kravchenko, S. V. and Sarachik, M. P. (2004). *Reports on Progress in Physics*, **67**, 1.
- Landau, L. D. (1957). Sov. Phys. JETP, 3, 920.
- Landau, L. D. (1959). Sov. Phys. JETP, 8, 70.
- Lee, Mark, Massey, J. G., Nguyen, V. L., and Shklovskii, B. I. (1999, Jul). *Phys. Rev. B*, **60**(3), 1582–1591.
- Lee, P. A. and Ramakrishnan, T. V. (1985, 4). Rev. Mod. Phys., 57(2), 287.
- Limelette, P., Georges, A., Jerome, D., Wzietek, P., Metcalf, P., and Honig, J.M. (2003a). Science, **302**, 89.
- Limelette, P., Wzietek, P., Florens, S., Georges, A., Costi, T. A., Pasquier, C., Jerome, D., Meziere, C., and Batail, P. (2003b). Phys. Rev. Lett., 91, 016401.
- M. A. Paalanen, R. N. Bhatt (1991). *Physica B*, **169**, 231.
- Maier, Thomas, Jarrell, Mark, Pruschke, Thomas, and Hettler, Matthias H. (2005, Oct). *Rev. Mod. Phys.*, **77**(3), 1027–1080.
- McGreevy, John (2010, Oct). *Physics*, **3**, 83.
- Mezard, M. and Monasson, R. (1994). Phys. Rev. B, 50, 7199–7202.
- Mézard, M., Parisi, G., and Virasoro, M. A. (1986). *Spin Glass theory and beyond*. World Scientific, Singapore.
- Mezard, M. and Young, A. P. (1992). Europhys. Lett., 18, 653-659.
- Millis, A. J. (1993). Phys. Rev. B, 48(10), 7183.

Milovanović, M., Sachdev, S., and Bhatt, R. N. (1989). Phys. Rev. Lett., 63, 82.

Miranda, E. and Dobrosavljević, V. (2001). Phys. Rev. Lett., 86, 264.

- Miranda, E. and Dobrosavljevic, V. (2005). Reports on Progress in Physics, 68, 2337.
- Moeller, G., Dobrosavljević, V., and Ruckenstein, A. E. (1999, Mar). *Phys. Rev.* B, **59**(10), 6846–6854.
- Mooij, J. H. (1973). Phys. Status Solidi A, 17, 521–530.
- Moreo, Adriana, Mayr, Matthias, Feiguin, Adrian, Yunoki, Seiji, and Dagotto, Elbio (2000). *Phys. Rev. Lett.*, **84**, 5568.
- Mott, N. F. (1949). Proc. Roy. Soc. (London), A197, 269.
- Mott, N. F. (1990). Metal-Insulator Transition. Taylor & Francis, London.
- Nelson, David R. and Pelcovits, Robert A. (1977, Sep). *Phys. Rev. B*, **16**(5), 2191–2199.
- Paalanen, M. A., Graebner, J. E., Bhatt, R. N., and Sachdev, S. (1998). *Phys. Rev. Lett.*, **61**, 597.
- Paalanen, M. A., Rosenbaum, T. F., Thomas, G. A., and Bhatt, R. N. (1982). Phys. Rev. Lett., 48, 1284.
- Pankov, S. and Dobrosavljević, V. (2005). Phys. Rev. Lett., 94, 046402.
- Parisi, G. and Sourlas, N. (1979). Phys. Rev. Lett., 43, 744.
- Park, H., Haule, K., and Kotliar, G. (2008, Oct). Phys. Rev. Lett., 101(18), 186403.
- Park, S., Hur, N., Guha, S., and Cheong, S. W. (2004). Phys. Rev. Lett., 93, 107207.
- Pastor, A. A. and Dobrosavljević, V. (1999). Phys. Rev. Lett., 83, 4642.
- Pastor, A. A., Dobrosavljević, V., and Horbach, M. L. (2001). *Phys. Rev. B*, **66**, 014413.
- Polyakov, A. M. (1975). *Physics Letters B*, **59**(1), 79 81.
- Popović, D., Fowler, A. B., and Washburn, S. (1997). Phys. Rev. Lett., 79, 1543.
- Qazilbash, M. M., Tripathi, A., Schafgans, A. A., Kim, Bong-Jun, Kim, Hyun-Tak, Cai, Zhonghou, Holt, M. V., Maser, J. M., Keilmann, F., Shpyrko, O. G., and Basov, D. N. (2011, Apr). *Phys. Rev. B*, 83(16), 165108.
- Radonjic, Milos M., Tanaskovic, D., Dobrosavljevic, V., and Haule, K. (2010, Feb). Phys. Rev. B, 81(7), 075118.
- Reichhardt, C., Reichhardt, C. J. Olson, and Bishop, A. R. (2005). EPL (Europhysics Letters), 72(3), 444.
- Richardella, Anthony, Roushan, Pedram, Mack, Shawn, Zhou, Brian, Huse, David A., Awschalom, David D., and Yazdani, Ali (2010). *Science*, **327**(5966), 665–669.
- Rosenbaum, T.F., Andres, K., Thomas, G.A., and Bhatt, R.N. (1980). Phys. Rev. Lett., 45, 1723.
- Sachdev, S. (2010, Oct). Phys. Rev. Lett., 105(15), 151602.
- Sachdev, S. (2011). *Quantum phase transitions, 2nd Edition*. Cambridge University Press, UK.
- Sachdev, S., Read, N., and Oppermann, R. (1995). Phys. Rev. B, 52, 10286.
- Sapienza, L., Thyrrestrup, H., Stobbe, S., Garcia, P. David, Smolka, S., and Lodahl, P. (2010). Science, **327**(5971), 1352–1355.
- Sarachik, M. P. (1995). In *Metal-Insulator Transitions Revisited* (ed. P. Edwards and C. N. R. Rao). Taylor and Francis, London.
- Schaffer, L. and Wegner, F. (1980). Phys. Rev. B, 38, 113.

60 References

Schmalian, J. and Wolynes, P. G. (2000). Phys. Rev. Lett., 85, 836.

- Shahar, D., Tsui, D. C., Shayegan, M., Shimshoni, E., and Sondhi, S. L. (1997, Jul). *Phys. Rev. Lett.*, **79**(3), 479–482.
- Shklovskii, B.I. and Efros, A.L. (1984). *Electronic properties of doped semiconductors*. Springer series in solid-state sciences. Springer-Verlag, BFsarachikerlin.
- Si, Qimiao, Rabello, Silvio, Ingersent, Kevin, and Smith, J. Lleweilun (2001, October). Nature, **413**(6858), 804–808.
- Siegrist, T., Jost, P., Volker, H., Woda, M., Merkelbach, P., Schlockermann, C., and Wuttig, M. (2011, March). Nat. Mater., 10(3), 202–208.
- Simonian, D., Kravchenko, S. V., and Sarachik, M. P. (1997, May). Phys. Rev. B, 55(20), R13421–R13423.
- Slater, J. C. (1934). Rev. Mod. Phys., 6, 209.
- Slater, J. C. (1951). Phys. Rev., 82, 538.
- Smith, J. L. and Si, Q. (2000). Phys. Rev. B, 61, 5184.
- Stauffer, D. and Aharoni, A. (1994). *Introduction to Percolation Theory* (2nd edn). Taylor & Francis, London.
- Stewart, G. R. (1984). Rev. Mod. Phys., 56, 755.
- Straley, J. P. (1977). Phys. Rev. B, 15, 5733.
- Tanaskovic, D., Dobrosavljevic, V., and Miranda, E. (2005). Physical Review Letters, 95, 167204.
- Tanasković, D., Miranda, E., and Dobrosavljević, V. (2004). Phys. Rev. B, 70, 205108.
- Tanatar, B. and Ceperley, D. M. (1989, Mar). Phys. Rev. B, **39**(8), 5005–5016.
- Terletska, H. and Dobrosavljević, V. (2011, May). Phys. Rev. Lett., 106(18), 186402.
- Terletska, H., Vučičević, J., Tanasković, D., and Dobrosavljević, V. (2011, Jul). *Phys. Rev. Lett.*, **107**(2), 026401.
- Thomas, G. A., Capizzi, M., DeRosa, F., Bhatt, R. N., and Rice, T. M. (1981, May). *Phys. Rev. B*, **23**(10), 5472–5494.
- Thouless, D. J. (1978). Journal of Physics C: Solid State Physics, 11(6), L189.
- Thouless, D. J., Anderson, P. W., and Palmer, R. G. (1977). *Philosophical Magazine*, **35**, 137.
- Ting, C. S., Lee, T. K., and Quinn, J. J. (1975, Apr). Phys. Rev. Lett., 34(14), 870–874.
- Vojta, Thomas (2006, 16 May). J. Phys. A: Math. Gen., 39, R143-R205.
- Vollhardt, D. (1984). Rev. Mod. Phys., 56, 99.
- Waintal, X. (2006, Feb). Phys. Rev. B, 73(7), 075417.
- Wegner, F. (1979). Phys. Rev. B, 35, 783.
- Wegner, Franz J. (1976). Zeitschrift fur Physik B Condensed Matter, 25, 327–337.
- Wen, Xiao-Gang (1992, Aug). Phys. Rev. B, 46(5), 2655–2662.
- Widom, B. (1965). J. Chem. Phys., 43(11), 3898.
- Wigner, E. (1934, Dec). Phys. Rev., 46(11), 1002–1011.
- Wilson, K. G. and Fisher, M. E. (1972, Jan). Phys. Rev. Lett., 28(4), 240–243.
- Wilson, K. G. and Kogut, J. (1975). *Physics Reports*, 12C(2).
- Yang, S.-X., Fotso, H., Su, S.-Q., Galanakis, D., Khatami, E., She, J.-H., Moreno, J., Zaanen, J., and Jarrell, M. (2011, Jan). *Phys. Rev. Lett.*, **106**(4), 047004.

References 61

Zala, G., Narozhny, B. N., and Aleiner, I. L. (2001). *Phys. Rev. B*, **64**, 214204(31). Zhang, Y. and Das Sarma, S. (2005, Jan). *Phys. Rev. B*, **71**(4), 045322.